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A Partial Differential Equation Approach to Modeling Simple Extension in Elastomers*

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1 Introduction

Elastomers, rubber–like polymers, are found in a vast array of engineering applications, ranging from traditional products such as tires to more modern applications, such as road bridge bearings that allow for thermal expansion of the bridge (see [9, 11, 12]). For these applications, rubber is combined with inactive fillers, frequently carbon black or silica, which enhance the physical properties to fit design specifications. As smart materials technology evolves, rubber is inevitably being considered for more advanced engineering roles. For example, a rubber rod with a matrix of embedded fiber optic cables might be used as a strain sensor, and an active vibration suppression device might be made by using piezoelectric or ferrous particles as fillers.

Many static models (see [10, 14, 15]) have been developed for elastomers, mainly based on either Rivlin's finite strain (FS) theory or on strain energy functions (SEFs). Both classes of models rely on use of the principal stretch ratios, the deformed length of a unit vector parallel to the principal axes (the axes of zero shear). Although some of these models have produced excellent fits to loading curves resulting from static testing, they do not address the loss of potential energy (hysteresis) inherent to elastomers, and hence cannot be used to fit the "loops" resulting from loading followed by unloading. The development of more sophisticated, dynamic models is critical to the design of elastomer components for use in dynamic conditions.

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Elastomers exhibit a number of complex behaviors, in addition to the nonlinear constitutive laws and hysteresis, that are essential elements of a successful dynamic model. Strong damping (loss of kinetic energy) is characteristic of these materials. The state of strain also depends in a non-trivial way on strain history, environmental temperature, rate of loading, and amount and type of filler. The nonlinear partial differential equation (PDE) model presented here for a slender rod with a tip mass undergoing simple extension includes damping, but not hysteresis. Models for hysteresis are currently being developed, and will eventually be used in conjunction with this PDE model.

Constitutive laws, arising either from a SEF or from Rivlin's finite strain formulation, can be used along with material independent force and moment balance derivations (the Timoshenko theory [8, 13]) as the basis of dynamic models. To illustrate this we take the simplest example: an isotropic, incompressible rubber-like rod (with a tip mass) under simple elongation with a finite applied stress in the principal axis direction $x_1 = x$, as seen in Figure 1.1. The position at any time t of the slice that was at location x in the unstrained body is designated by $u(t,x)$. The engineering stress is given by

$$S = \frac{E}{3}\left(\frac{\partial u}{\partial x} + \hat{g}\left(\frac{\partial u}{\partial x}\right)\right) + C_D \frac{\partial \dot{u}}{\partial x}\ , \tag{1.1}$$

where a Kelvin-Voigt damping term has been included as a first attempt at modeling damping (see [4, 6] for more details). For a Hookean material $\hat{g}(\xi) = 2\xi$, and for a neo-Hookean material $\hat{g}(\xi) = 1 - (1+\xi)^{-2}$ (see [4]). Here E is a generalized modulus of elasticity and we note these formulations are restricted by the physical constraint $\frac{\partial u}{\partial x} > -1$. Substituting the engineering stress (1.1) into the Timoshenko theory for longitudinal vibrations of a rubber bar with a tip mass, we obtain the nonlinear partial differential equation initial boundary value problem (PDE IBVP)

$$\begin{aligned}
\rho A_c \ddot{u} + \gamma \dot{u} - \frac{\partial}{\partial x}\left(\frac{EA_c}{3}\left(\frac{\partial u}{\partial x} + \hat{g}\left(\frac{\partial u}{\partial x}\right)\right) + A_c C_D \frac{\partial \dot{u}}{\partial x}\right) &= 0 \quad 0 < x < l \\
M\ddot{u}(t,l) = -\left(\frac{A_c E}{3}\left(\frac{\partial u}{\partial x} + \hat{g}\left(\frac{\partial u}{\partial x}\right)\right) + C_D A_c \frac{\partial \dot{u}}{\partial x}\right)\Big|_{x=l} &+ \quad F(t) + Mg \\
u(t,0) = 0 \quad , \quad u(0,x) = \Delta(x) \quad , \quad \dot{u}(0,x) = 0 &
\end{aligned} \tag{1.2}$$

for the dynamic longitudinal displacement of a rod in extension. In this case ρ = mass density, $F(t)$ = applied external force, A_c is the cross sectional area, M is the tip mass, g is the gravitational constant, and γ is the air damping coefficient.

In general, one does not expect the initial boundary value problem associated with (1.2) to have a classical (smooth) solution. For this reason,

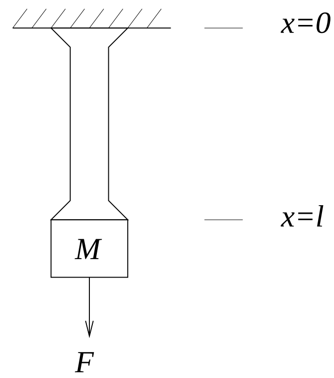

Figure 1.1: Rod with tip mass under tension

it is useful to consider the abstract form of (1.2)

$$\begin{aligned} \ddot{y}(t) &+ A_2\dot{y}(t) + A_1 y(t) + \mathcal{N}^* G(\mathcal{N}y) = F(t) \text{ in } V^* \\ y(0) &= y_0, \quad \dot{y}(0) = y_1 \end{aligned} \tag{1.3}$$

in the state $y = (u(t,l), u(t,\cdot))$. The pivot space $H = \mathbb{R}^1 \times L_2(0,l)$ is endowed with the inner product

$$\langle(\eta,\phi),(\mu,\psi)\rangle_H = M\eta\mu + \langle\rho A_c \phi, \psi\rangle_0$$

where $\langle\cdot,\cdot\rangle_0$ is the usual L_2 inner product on the interval $(0,l)$, and the state space $V = \{(\eta,\phi) : \phi \in H^1_L(0,l), \eta = \phi(l)\}$ has inner product

$$\langle(\eta,\phi),(\mu,\psi)\rangle_V = M\eta\mu + \langle\rho A_c \frac{\partial\phi}{\partial x}, \frac{\partial\psi}{\partial x}\rangle_0 \ .$$

The spaces V and H form a Gelfand triple $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ with pivot space H and duality paring $\langle\cdot,\cdot\rangle_{V^*,V}$.

The linear operator

$$A_1 \in \mathcal{L}(V,V^*), A_1\Psi = \left(\frac{EA_c}{3}\frac{\partial\psi}{\partial x}(l), -\frac{\partial}{\partial x}\left(\frac{EA_c}{3}\frac{\partial\psi}{\partial x}\right)\right)$$

where the derivatives are in the distributional sense, has associated sesquilinear form

$$\sigma_1(\Phi,\Psi) \equiv \langle A_1\Phi,\Psi\rangle_{V^*,V} = \langle\frac{EA_c}{3}\frac{\partial\phi}{\partial x}, \frac{\partial\psi}{\partial x}\rangle_0 \ ,$$

and the linear operator $A_2\Psi = (C_D A_c \frac{\partial\psi}{\partial x}(l), -\frac{\partial}{\partial x}\left(C_D A_c \frac{\partial\psi}{\partial x}\right) + \gamma\psi)$, where $A_2 \in \mathcal{L}(V,V^*)$, with derivatives again in the distributional sense, has associated sesquilinear form

$$\sigma_2(\Phi,\Psi) \equiv \langle A_2\Phi,\Psi\rangle_{V^*,V} = \langle C_D A_c \frac{\partial\phi}{\partial x}, \frac{\partial\psi}{\partial x}\rangle_0 + \langle\gamma\phi,\psi\rangle_0 \ .$$

The linear operator $\mathcal{N} \in \mathcal{L}(V,H)$ is given by

$$\mathcal{N}\Phi = (\phi(l), \frac{\partial \phi}{\partial x}) \ ,$$

and its adjoint $\mathcal{N}^* \in \mathcal{L}(H,V^*)$. The function G is given by

$$G(\eta,\phi) = (0, \frac{E}{3\rho}\hat{g}(\phi)) \ ,$$

where $\hat{g}(\cdot)$ comes from the constitutive law (1.1), and the function

$$F(t) = (f(t) + Mg, 0) \ ,$$

where $F(t)$ is the applied external force. Equation (1.3) fits into the Banks, Gilliam, Shubov framework [1], and thus the problem is well-posed if appropriate monotonicity and smoothness conditions are satisfied by G. We will assume that these conditions hold for the remainder of this paper, and defer the details of the arguments to a later paper.

Having established well-posedness, we proceed to Section 2 and a discussion of numerical methods for approximating the solution of (1.3). In Section 3 we will present an inverse methodology, along with numerical results. In Section 4 we summarize current and future work to augment and refine the modeling efforts.

2 Numerical Approximation

For a given G satisfying the Banks, Gilliam, Shubov well-posedness criteria in [1], equation (1.3) may be written using the weak, or variational, form: for every $\Phi \in V$,

$$\langle \ddot{y}, \Phi \rangle_{V^*,V} + \sigma_2(\dot{y}, \Phi) + \sigma_1(y, \Phi) + \mathcal{N}^*(G(\mathcal{N}y))\Phi = \langle F(t), \Phi \rangle_{V^*,V}, \quad (2.4)$$

and Galerkin approximations may be defined. For any set of linearly independent basis functions which form a total set in V, the Galerkin approximates are guaranteed to converge (see [1]). We will employ a Galerkin method using linear splines (appropriate in the $H^1(0,l)$ setting) for the basis elements in the spatial discretization. The system with tip mass is stiff, and for that reason a stiff integrator, such as Gear's BDF method, should be used in time. We seek an approximate solution to (1.3) of the form

$$u^N(t,x) = \sum_{j=1}^{N} \alpha_j^N(t) B_j^N(x)$$

where the basis functions $B_j^N(x) = (L_j^N(l), L_j^N(x))$, the L_j^N are the usual linear splines with grid spacing $h = l/N$, and the coefficients α_j^N are unknown functions of time. The boundary condition $u(t,0) = 0$ is treated as an essential boundary condition and is imposed directly on each of the basis elements L_j^N.

In order to use Gear's method, (1.3) must be written as a first order system in time. To accomplish this, rewrite the weak formulation (2.4) by choosing $w_1 = y$, thus

$$\dot{w}_1 = w_2 \tag{2.5}$$

and for each $\Phi \in V$

$$\langle \dot{w}_2, \Phi \rangle_{V^*,V} = -\sigma_2(w_2, \Phi) - \sigma_1(w_1, \Phi) - \mathcal{N}^*(G(\mathcal{N} w_1))\Phi + \langle F(t), \Phi \rangle_{V^*,V} \; . \tag{2.6}$$

The approximate solutions of (2.5) and (2.6) are given by (for ease in notation, we henceforth drop the superscript N on the elements $B_j = B_j^N$, $L_j = L_j^N$, and coefficients $\alpha_j = \alpha_j^N, \beta_j = \beta_j^N$)

$$w_1^N(t,x) = \sum_{j=1}^N \alpha_j(t) B_j(x) \; ,$$

and

$$w_2^N(t,x) = \sum_{j=1}^N \beta_j(t) B_j(x) \; .$$

Substituting the approximate solutions into (2.5) we obtain equations

$$\dot{\alpha}_j(t) = \beta_j(t) \; , \; j = 1 \, , \ldots \, , N \; .$$

Substituting the approximate solutions into (2.6), and choosing $\Phi = B_k$, we find

$$\begin{aligned} \langle \sum_{j=1}^N \dot{\beta}_j B_j, B_k \rangle_{V^*,V} &= -\sigma_2(\sum_{j=1}^N \beta_j B_j, B_k) - \sigma_1(\sum_{j=1}^N \alpha_j B_j, B_k) \\ &- \mathcal{N}^*(G(\mathcal{N} \sum_{j=1}^N \alpha_j B_j)) B_k + \langle F(t), B_k \rangle_{V^*,V} . \end{aligned} \tag{2.7}$$

For fixed t the discrete system is achieved by allowing k to vary between 1 and N. The following notation is necessary to write down the discrete system. Define the (symmetric) tridiagonal mass matrix $\mathbf{M}$ with diagonal entries

$$\begin{aligned} [\mathbf{M}_{i,i}] &= \int_0^l \rho A_c L_i^2 dx \; , \;\; i < N \; , \\ [\mathbf{M}_{N,N}] &= M + \int_0^l \rho A_c L_N^2 dx \end{aligned}$$

and sub- and super-diagonal entries

$$[\mathbf{M}_{i+1,i}] = [\mathbf{M}_{i,i+1}] = \int_0^l \rho A_c L_i L_{i+1} dx \ .$$

The (symmetric) tridiagonal stiffness matrix $\mathbf{K}$ has entries

$$[\mathbf{K}_{i,j}] = -\int_0^l \frac{EA_c}{3} L_i' L_j' dx$$

and the (symmetric) tridiagonal damping matrix $\mathbf{D}$ has entries

$$[\mathbf{D}_{i,j}] = -\int_0^l C_D A_c L_i' L_j' dx \ \ -\gamma \int_0^l L_i L_j dx \ .$$

The vector $\vec{G}$ is given by

$$\vec{G}_i = -\int_0^l L_i' \frac{A_c E}{3} \hat{g}(\sum_{j=1}^N \alpha_j L_j') dx \ ,$$

and the vector $\vec{F} = (0, \ldots, 0, F(t) + Mg)^T$. The first order discrete system is then

$$\begin{bmatrix} I & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\vec{\alpha}} \\ \dot{\vec{\beta}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \vec{G}(\vec{\alpha}) \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \vec{F}(t) \end{bmatrix} ,$$

$$\begin{bmatrix} \vec{\alpha}(0) \\ \vec{\beta}(0) \end{bmatrix} = \begin{bmatrix} \vec{y_0} \\ \vec{y_1} \end{bmatrix} .$$

3 Approximation of Constitutive Laws

The standard SEF models used to find an exact form for the engineering stress (1.1) are inadequate in describing most elastomers. Experimental results can be used to establish more accurate (and more general) estimates of the nonlinearity $\hat{g}$ in $\tilde{g} = \frac{E}{3}(\frac{\partial u}{\partial x} + \hat{g}(\frac{\partial u}{\partial x}))$. These approximate constitutive laws should not be expected to admit a SEF as a function of either the strain invariants or the stretch ratios. For static problems, comparisons with SEF methods (which are widely accepted static models in industry) can be made by using the (approximate) SEF to derive the expected stress-strain relationship, and comparing results in the stress-strain (or, equivalently, the load-deflection) plane. The static form of the PDE has been used to approximate constitutive laws which compare favorably to results from a standard SEF package (see [6, 5]). This is an important foundation, since there are no existing packages to calculate the dynamic constitutive relationship.

To determine the nonlinearity $\hat{g}(\xi)$, one has, for a given input $F(t)$, observations z_i which are proportional to the strain $\frac{\partial u}{\partial x}(t_i, 0)$ at the fixed end. The estimation problem of interest consists of minimizing over $(\hat{g}, C_D, \gamma)$ in some admissible class $\mathcal{G} \times \mathbb{R}^2_+$

$$J(\hat{g}, C_D, \gamma) = \sum_{i=1}^{Nt} |z_i - \mu \frac{\partial u}{\partial x}(t_i, 0; \hat{g}, C_D, \gamma)|^2 \tag{3.8}$$

where $y = (u(t,l), u(t,\cdot))$ is the solution of (2.4) corresponding to $\hat{g}, C_D$, and γ. One may also be fortunate enough (although this is not the case for the experimental results that we are about to present) to have observations $\hat{u}_i$ of deformations $u(t_i, \bar{x}; \hat{g}, C_D, \gamma)$ at some point $x = \bar{x}$, $0 < \bar{x} < l$, and then the optimization criterion (3.8) can be modified accordingly.

Because the problem involving (3.8) and (2.4) is an infinite dimensional problem in both state and parameter space, finite dimensional approximations must be made for computational purposes. For state approximations, we use the Galerkin techniques discussed in Section 2, along with Gear's BDF method for time integration. For parameter space discretization, one may use a finite dimensional parameterization or representation of $\hat{g}$. One of the simplest methods is to approximate $\hat{g}$ using piecewise linear elements,

$$\hat{g}_M(x) = \sum_{j=1}^{M} c_j \eta_j(x) \ .$$

The least squares spline inverse problem (LSSIP) is then equivalent to: find $\vec{c} \in \mathbb{R}^{M+2}$ (with $c_{M+1} = C_D$ and $c_{M+2} = \gamma$) minimizing

$$J(\vec{c}) = \sum_{i=1}^{Nt} |z_i - \mu \frac{\partial u^N}{\partial x}(t_i, 0; \vec{c})|^2 \ .$$

The majority of the dynamic tests currently performed on elastomers are cyclic, with either sinusoidal applied forces or forced sinusoidal end displacement. Because our primary goal is to identify the constitutive relationship, we used numerical simulations to predict the results of various dynamic tests. As reported in [6, 7], the difference between a Hookean material and a neo–Hookean material was most readily seen under free vibration testing conditions. An added benefit of free vibration testing is that it provides a natural setting for the study of damping properties. We have chosen to use free release tests, which result in larger deformations than hammer hit tests.

Compression leads to nontrivial shear, and should be avoided to obtain the best results when investigating simple tension or elongation. To accomplish this, a tip mass was attached to the end of the rod. In order to minimize the contribution of hysteresis, which is not included in our current model, a slender rod composed of *unfilled* natural rubber was used, as depicted in Figure 3.2. The rod length was $l = 5.4356$ cm, with flange height 0.3048 cm, inner diameter ID = 0.4572 cm, outer diameter OD = 1.905 cm, and the metal tabs were 1.27 cm high. The frame (which was used both as a mass and as a housing to protect an accelerometer) had mass 262.7 g, and the sample (including the bonded metal tabs) had mass 52.1 g. Data was collected using three Hewlett Packard digital analyzers. The purpose of the top accelerometer was to verify that we physically obtained a reasonable approximation to a “clamped end” boundary condition, while the accelerometer in the frame was used to corroborate data obtained from the load cell.

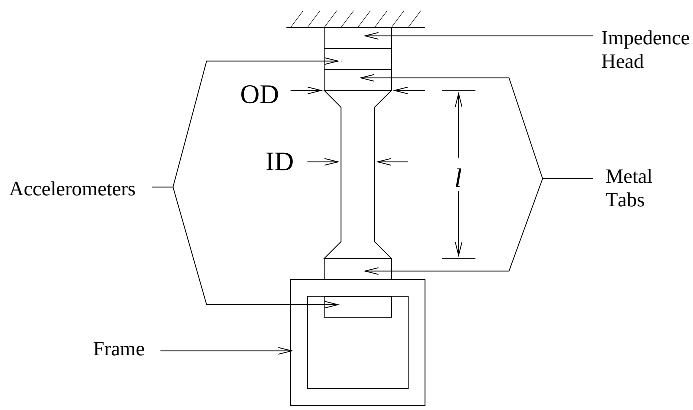

Figure 3.2: Experimental setup for rod under tension

For the free release experiments, the assembly was lifted so that the rod was at its natural length (i.e., no compression or extension). The support was then removed, allowing the mass to fall freely. The maximum dynamic strain achieved in this test was approximately 34%. As seen in Figure 3.3, the 8-term piecewise linear constitutive approximation provides a model that gives a close fit to the data in both the time and frequency domains. Figure 3.4 depicts a linear approximation and an 8-term piecewise linear approximation to the the static constitutive relationship.

4 Summary

The methods presented in this paper provide a first insight to the dynamic mechanical behavior of elastomers. The Kelvin Voigt term used in the model represents our initial effort to model damping. While the results we obtained were acceptable for unfilled natural rubber, most applications

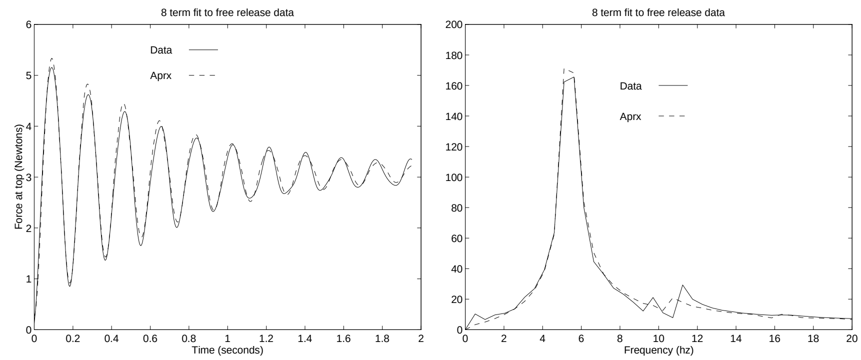

Figure 3.3: 8-term fit, air and Kelvin Voigt damping for free release data in time and frequency domains

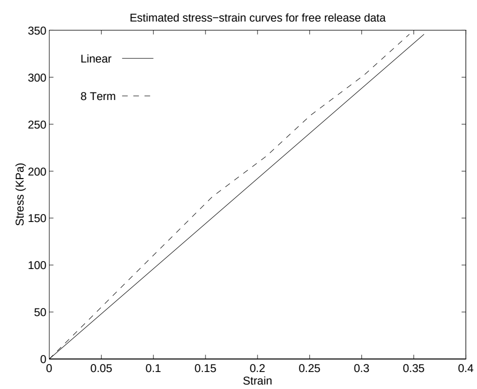

Figure 3.4: Static linear stress-strain relationship vs 8-term nonlinear stress-strain relationship

require filled elastomers, which are often more highly damped. Further studies should be conducted using a variety of damping models and more highly damped materials.

Hysteresis is highly significant in a filled rubber, and must be included in any successful model. The modeling implications of this are fairly obvious: one cannot represent the stress-strain constitutive law by a simple nonlinearity $\tilde{g}$. Instead, a family of stress-strain laws $\{\tilde{g}\}$, along with some type of memory mechanism must be used. Experimental data has been collected, and modeling efforts based on generalized hysteresis measure ideas (developed for shape memory alloy actuators in [2, 3]) have begun.

Because shear is significant in elastomers, shear models must be developed in order to predict more general deformations. Our initial attempts to develop models for elastomers in generalized simple shear can be found in [4, 5]. We are in the process of using numerical simulations to design experiments which will be most effective in identifying the constitutive relationship, as we did in the modeling efforts for simple extension.

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