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A Control Theoretic Model of the Muscular Actions in Human Head-Eye Coordination*

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Abstract

In this article we investigate the problem of how to model and control the combined motion of the human head and eye. We develop a model of the muscles, based on a simplified physical model and an assumption that the muscles can be modeled as damped springs with a second order linear dynamics. We then find control laws that both make the combined pupil-movement follow a given trajectory, and make the separate head and eye trajectories three times continuously derivable. Our controls also make the energy produced in the movement small, since we believe that to be a reasonable, physical control-criterion.

1 Introduction

In this article, two somewhat separate questions are being discussed, and the first one concerns finding a mathematical model of the combined, horizontally rotational movements of the human head and eye. Therefore we devote Section 2 to finding systems of differential equations for describing the movements, based on simplified physical models of the muscular configurations in the neck and the eye respectively. However, it must be stressed that even though we use simplified models, our aim is to come up with a physically feasible model for the human muscular actions. This model could be of some interest in robotics, but our primary goal is that it will help us understand the dynamics of the actual muscles.

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The next task, when it comes to finding a mathematical model, is to link the two separate systems, constituted by the head and the eye respectively, together, so that we can move on to the next major problem investigated in this article; How do we combine the movements of the head and the eye in order to follow a moving object with a given, known trajectory, at a constant distance from the head? This question is discussed in Section 3, where control laws are developed for activating the neck and the eye muscles in such a way that the pupil follows the desired trajectory, at the same time as both the head and eye trajectories, viewed separately, are three times continuously differentiable.

The reason for investigating the known trajectory case is that even though we in practice do not know the trajectory, we believe that since the problem involves four actuators, one for each muscle, the control of the overall switching system is interesting for it's own sake. We also hope that we further on are going to be able to make predictions of the observed object's trajectory, and then base the tracking on these predictions. This can be done using the same strategies as those suggested in this article.

But any control laws that accomplish this will not do. Therefore we dedicate Section 4 to the optimal control problem, and the results are then discussed in Section 5 followed by a presentation of some of the graphs produced in the different simulations that were conducted.

2 Head and Eye Rotation

2.1 Dynamics of head rotation

Human head movements are controlled by more than 20 pairs of muscles that link the skull, spinal column and shoulder girdle in a complex variety of configurations.

In this paper, concern is only given to the muscles that control the horizontal rotation of the head. This task is mainly handled by five muscles on each side of the body. Three of these five muscles, the *semispinals*, the *erector spinae* and the *multifidus* are located on the back side of the spine, and they only function as assistant movers in the rotation movement. The other two are the *sternocleidomastoid* and the *splenius*.

What we want to do in this paper is to model the complex behavior of all those muscles, primarily the sternocleidomastoids and the splenius, in such a way that the rotation of the head is given account for in a simple way. Therefore we chose to model the muscles as just one pair of muscles, conducting the same actions as all the five muscles together. This is because we are more interested in the principles of the controls behind the muscular contractions, than in finding an exact muscular model at the price of clarity.

We chose to model these muscles as damped springs with a second order linear dynamics of the form

$$\ddot{x} = -k(x - L) - g\dot{x} + v(t), \tag{1}$$

where L and x are the lengths of the unstretched and the stretched spring respectively, and k and g are frequency and damping parameters of the spring. A controller, $v(t)$, is added to the spring, and the control term is only added to one of the two muscles at a time, since only one muscle is active when the head is rotating.

This might not be the best way to model the muscles, since an actual muscle is much more complicated than a spring and has a somewhat nonlinear structure, but this model suffices to simplify the problem, and works satisfactory for the purpose of mathematical calculations [13]. This linear model is also a pretty good approximation of the muscles for small muscular contractions, where nonlinear terms do not contribute that much to the dynamics.

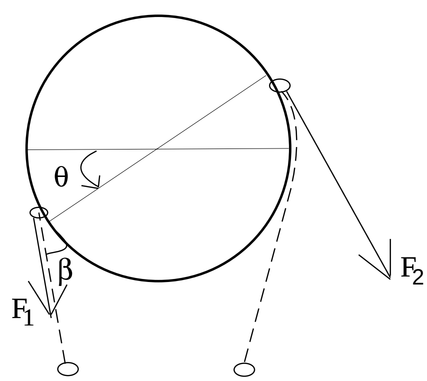


Figure 1: The two forces producing a rotation of the head.

If the angle θ is chosen to be the system state variable, as seen in Figure 1, then we get

$$\ddot{\theta} = \frac{R}{I}(F_1 \cos\beta - F_2), \tag{2}$$

where I is the moment of inertia of the disc that is being rotated, since the angular acceleration is given by the torque, produced by the two tangential forces $F_1 \cos\beta$ and F_2. If we now let x_1 and x_2 be the lengths of the left and the right spring respectively and consider the fact that we now have two springs affecting the lengths simultaneously, we get

$$\ddot{x}_2 = k(\cos\beta(x_1 - L) - (x_2 - L)) + g(\cos\beta\dot{x}_1 - \dot{x}_2) - \cos\beta v_1(t) + v_2(t). \tag{3}$$

The Law of Cosines directly gives

$$\begin{aligned} x_1^2 &= L^2 + (2R\sin\frac{\theta}{2})^2 - 2L(2R\sin\frac{\theta}{2})\cos\frac{\theta}{2} \\ &= 4R^2\sin^2\frac{\theta}{2} + L^2 - 2LR\sin\theta, \end{aligned} \tag{4}$$

and we also know that

$$x_2 = R\theta + L \Rightarrow \ddot{\theta} = \frac{\ddot{x}_2}{R}. \tag{5}$$

This gives us

$$\ddot{\theta} = \frac{1}{R}\{(k(x_1 - L) + g\dot{x}_1 - v_1(t))\cos\beta + k(L - x_2) - g\dot{x}_2 + v_2(t)\}, \tag{6}$$

and after some calculations, we finally end up with a system on the form

$$\ddot{\theta} = f(\theta, \dot{\theta}) + g(\theta)v_1(t) + \frac{1}{R}v_2(t), \tag{7}$$

where

$$f(\theta, \dot{\theta}) = -(g\dot{\theta} + k\theta) + kLg(\theta) + h(\theta)[k + \frac{gR(R\sin\theta - L\cos\theta)\dot{\theta}}{L^2 + 4R^2\sin^2\frac{\theta}{2} - 2LR\sin\theta}], \tag{8}$$

$$g(\theta) = -h(\theta)\frac{1}{\sqrt{L^2 + 4R^2\sin^2\frac{\theta}{2} - 2LR\sin\theta}}, \tag{9}$$

$$h(\theta) = \frac{1}{R}(L\cos\theta - R\sin\theta) \tag{10}$$

and

$$v_1(t)v_2(t) = 0 \quad \forall t. \tag{11}$$

If we now allow θ to be negative, symmetry considerations directly gives that the total system is

$$\ddot{\theta} = \mathrm{sign}(\theta)f(|\theta|, \mathrm{sign}(\theta)\dot{\theta}) + u(\theta)v(t), \tag{12}$$

where

$$u(\theta) = \begin{cases} \mathrm{sign}(\theta)g(|\theta|) & \text{if } \mathrm{sign}(\theta) = \mathrm{sign}(\dot{\theta}) \\ \mathrm{sign}(\theta)\frac{1}{R} & \text{if } \mathrm{sign}(\theta) \neq \mathrm{sign}(\dot{\theta}) \end{cases} \tag{13}$$

and

$$v(t) = \begin{cases} v_1(t) & \text{if } \dot{\theta} > 0 \\ v_2(t) & \text{if } \dot{\theta} < 0. \end{cases} \tag{14}$$

This last condition (equation 14) gives us a physically inspired description of when we are to switch between actively controlling one muscle to the other.

2.2 Dynamics of occular motion

Since the main interest in this paper lies on finding controls that make the eyes and the neck act together in a satisfying way, only monocular vision is being studied. Monocular vision means that we only use one eye, located in the middle of the head, between the actual eyes of a human being, but this is not a serious restriction since the binocular case can be derived in almost the same way as the monocular case [13].

The *external* and *internal recti*, the muscles behind the rotation, both attach on the so-called Annulus of Zinn, behind the eye, and they also attach rather high up on the eye itself, which makes the modeling a bit easier than in the head case, since the geometry is simplified by the fact that the forces, produced by the two muscles, can be assumed to always be tangential to the eye itself.

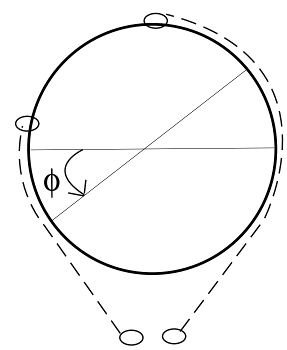


Figure 2: The geometry of the external and internal recti.

In almost the same way as in the neck case, the forces that produce the rotation of the eye are generated by

$$\ddot{x}_2 = k(x_1 - l - (x_2 - l)) + g(\dot{x}_1 - \dot{x}_2) - v_1(t) + v_2(t). \tag{15}$$

We also know that

$$x_1 = l - r\phi \tag{16}$$

$$x_2 = l + r\phi \Rightarrow \ddot{\phi} = \frac{\ddot{x}_2}{r}. \tag{17}$$

After some calculations, this gives us the total system as

$$\ddot{\phi} = -2(g\dot{\phi} + k\phi) - \text{sign}(\dot{\phi})\frac{1}{r}v(t), \tag{18}$$

with

$$v(t) = \begin{cases} v_1(t) & \text{if } \dot{\phi} > 0 \\ v_2(t) & \text{if } \dot{\phi} < 0, \end{cases} \tag{19}$$

as in the head case.

2.3 The combined dynamics

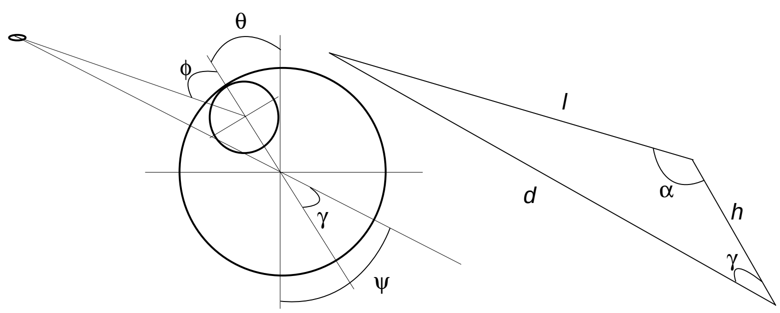


Figure 3: The geometry behind the combined movement.

So if we return to our initial problem; How do we combine the movements of the head and the eye in order to follow an object with a given trajectory at a constant distance from the head? If $\psi(t)$ is the tracked object's trajectory, Figure 3 directly gives the equation

$$\psi(t) = \theta(t) + \gamma(t). \tag{20}$$

The Law of Cosines, gives us

$$\begin{aligned} d^2 &= h^2 + l^2 - 2hl\cos\alpha = h^2 + l^2 + 2hl\cos\phi, & (21) \\ l^2 &= d^2 + h^2 - 2dh\cos\gamma & (22) \end{aligned}$$

and therefore

$$\phi = \operatorname{sign}(\gamma)\arccos\left(\frac{d\cos(\gamma) - h}{\sqrt{d^2 + h^2 - 2dh\cos(\gamma)}}\right), \tag{23}$$

where $\gamma = \psi - \theta$. Now let

$$\eta(\gamma) = \frac{d\cos\gamma - h}{\sqrt{d^2 + h^2 - 2hd\cos\gamma}}. \tag{24}$$

We then have

$$\phi = \operatorname{sign}(\gamma)\arccos\eta(\gamma), \tag{25}$$

$$\dot{\phi} = -\text{sign}(\gamma)\frac{1}{\sqrt{1-\eta(\gamma)^2}}\frac{d\eta(\gamma)}{d\gamma}\dot{\gamma}, \tag{26}$$

$$\ddot{\phi} = -\text{sign}(\gamma)\frac{1}{\sqrt{1-\eta(\gamma)^2}}\left(\frac{d^2\eta(\gamma)}{d\gamma^2}\dot{\gamma}^2 + \frac{d\eta(\gamma)}{d\gamma}\ddot{\gamma} + (\frac{d\eta(\gamma)}{d\gamma}\gamma)^2\frac{\eta(\gamma)}{1-\eta(\gamma)^2}\right). \tag{27}$$

This can be stated in a more compact form as

$$\ddot{\phi} = F(\theta, \psi, \dot{\theta}, \dot{\psi}, \ddot{\theta}, \ddot{\psi}), \tag{28}$$

but from equation 18 we still have the almost linear equation

$$\ddot{\phi} = -2(g\dot{\phi} + k\phi) - \text{sign}(\dot{\phi})\frac{1}{r}v(t). \tag{29}$$

Combining these two equations makes it possible to impose a control on $\ddot{\theta}$, and then let the control on $\ddot{\phi}$ be given automatically as

$$v_{eye}(t) = -\text{sign}(\dot{\phi})r[F(\theta, \psi, \dot{\theta}, \dot{\psi}, \ddot{\theta}, \ddot{\psi}) + 2(g\dot{\phi} + k\phi)]. \tag{30}$$

This way of letting the main tracking be done by the eye is a product of the so-called *occulocentric view*. This means that the main tracking is performed by the eye, while the head is just moving in a general way, as seen in the next section. This approach is a rather reasonable one, since the fast *saccadic* movements of the eye make the eye better suited for following fast movements than the head [8].

3 Control Laws

Now that we have a model for the combined process of activating both the muscles of the neck and of the eye, the next task is to find the control laws. We want the pupil to follow a smooth trajectory, and in order to accomplish this, we need to find controls that make the pupil movement both smooth and completely determined by the tracked object's position. We also want the two separate movements, those of the head and those of the eye, to be continuously derivable at least three times. This smoothness constraint is given more or less by the fact that we want to control models of actual muscles, whose position, velocity and acceleration appear to be continuously differentiable functions of time.

But this constraint is not enough. We do not only want to find any control law that does what we want, we want to find one that does it well. Therefore we need a criterion by which we can determine how good any given solution is. A reasonable approach is to try to minimize the energy produced in the movement, since nature has a tendency towards energy minimization. Since the mass of the head, M, is so much larger than the mass of the eye, m, one criterion for finding our control could be that it should make the angular acceleration of the head as small as possible. This would make the energy, given by the torque, small since

$$E = \int_{\theta_0}^{\theta_f} |N\, d\theta| = \int_{\theta_0}^{\theta_f} |\ddot{\theta} I\, d\theta|. \tag{31}$$

In order to accomplish this, we divide the trajectory of the head into subparts, where in some parts the head accelerates, and in others the angular acceleration is zero. This is because one obvious control that makes $|\ddot{\theta}|$ small is the one that makes $\ddot{\theta} = 0$. Therefore we want the major part of the trajectory to be of this type.

If we assume that we start at θ_0 and stop at θ_f, what we want is the following scenario:

$$\begin{aligned} \dot{\theta}(t) &= \frac{\theta(t) - \theta_f}{t - t_f} = \text{const} \quad t \in [t_0, t_f], & (32) \\ \ddot{\theta}(t) &= 0 \qquad\qquad\qquad\quad t \in [t_0, t_f]. & (33) \end{aligned}$$

If we recall equation 12 - 14, we directly see that

$$\begin{aligned} 0 &= \text{sign}(\theta) f(|\theta|, \text{sign}(\theta)\dot{\theta}) + u(\theta) v_{lin}(t) \\ v_{lin}(t) &= -\frac{\text{sign}(\theta) f(|\theta|, \text{sign}(\theta)\dot{\theta})}{u(\theta)}. \end{aligned} \tag{34}$$

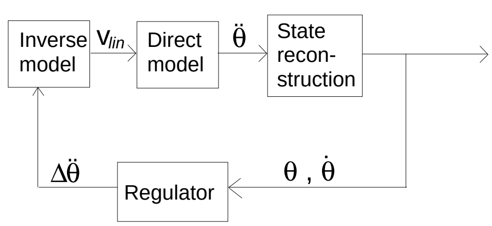


Figure 4: Block chart for the zero acceleration case.

The general idea can be seen in Figure 4, and we chose to use a feedback on the form

$$\Delta\ddot{\theta} = C\left(\frac{\theta(t) - \theta_f}{t - t_f} - \dot{\theta}(t)\right). \tag{35}$$

This linear approach is unfortunately not enough. First of all, we assume that we start following the object when the head and the eye both are at rest at some fixed angle, and therefore we need to find controls that can accelerate the systems up to some suitable velocity when the tracking is initiated. Secondly, when the followed trajectories are not as well behaved as $\psi(t) = \psi_0 - (t - t_0)$, we have to take into account that that the eye may rotate out of bound if no modification of the head's zero acceleration trajectory is being made. These two cases show that we need to be able to accelerate the head in a controlled way in some situations.

Inspired by the feed forward control system concept in [16], the general idea behind the control laws we chose to use, can be illustrated by the block chart in Figure 5.

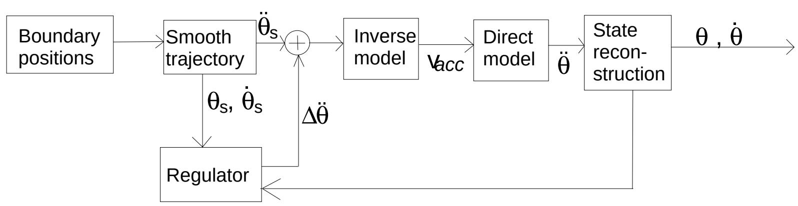


Figure 5: Block chart for the feed forward acceleration case.

Each period of acceleration has a duration of Δt, and it starts at t_a and ends at t_b ($\Delta t = t_b - t_a$), and if we use a polynomial for describing the accelerations, which, for calculation reasons, is a good choice, we need a polynomial for describing $\theta(t)$ with a degree of at least seven. This is because we need eight coefficients since we have eight continuity conditions that need to be fulfilled (four conditions at t_a, and four at t_b). If we let

$$\begin{aligned} \theta(t) &= \frac{1}{42}(t-t_a)^7 C_1 + \frac{1}{30}(t-t_a)^6 C_2 + \frac{1}{20}(t-t_a)^5 C_3 + \frac{1}{12}(t-t_a)^4 C_4 \\ &+ \frac{1}{6}(t-t_a)^3 C_5 + \frac{1}{2}(t-t_a)^2 C_6 + (t-t_a)C_7 + C_8, \end{aligned} \tag{36}$$

we get the following, well-defined equation system:

$$T = \begin{pmatrix} 5\Delta t^4 & 4\Delta t^3 & 3\Delta t^2 & 2\Delta t \\ \Delta t^5 & \Delta t^4 & \Delta t^3 & \Delta t^2 \\ \frac{1}{6}\Delta t^6 & \frac{1}{5}\Delta t^5 & \frac{1}{4}\Delta t^4 & \frac{1}{3}\Delta t^3 \\ \frac{1}{42}\Delta t^7 & \frac{1}{30}\Delta t^6 & \frac{1}{20}\Delta t^5 & \frac{1}{12}\Delta t^4 \end{pmatrix}, \tag{37}$$

$$C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}, \tag{38}$$

$$D = \begin{pmatrix} \frac{d^3\theta(t_b)}{dt^3} - \frac{d^3\theta(t_a)}{dt^3} \\ \ddot{\theta}(t_b) - \ddot{\theta}(t_a) - \Delta t \frac{d^3\theta(t_a)}{dt^3} \\ \dot{\theta}(t_b) - \dot{\theta}(t_a) - \Delta t \ddot{\theta}(t_a) - \frac{1}{2}\Delta t^2 \frac{d^3\theta(t_a)}{dt^3} \\ \theta(t_b) - \theta(t_a) - \Delta t \dot{\theta}(t_a) - \frac{1}{2}\Delta t^2 \ddot{\theta}(t_a) - \frac{1}{6}\Delta t^3 \frac{d^3\theta(t_a)}{dt^3} \end{pmatrix} \tag{39}$$

and

$$TC = D. \tag{40}$$

This gives us all we need to know, and if we want to find the control producing this trajectory, we simply use the inverse for $\ddot{\theta}$:

$$v_{acc}(t) = \frac{\ddot{\theta}_{poly} + \Delta\ddot{\theta} - \text{sign}(\theta) f(|\theta|, \text{sign}(\theta)\dot{\theta})}{u(\theta)}, \tag{41}$$

where $\ddot{\theta}_{poly}$ is the desired trajectory, and $\Delta\ddot{\theta}$ is the linear feedback. We chose to model the feedback on the form

$$\Delta\ddot{\theta}(t) = C_1(\theta_{poly}(t) - \theta_{actual}(t)) + C_2(\dot{\theta}_{poly}(t) - \dot{\theta}_{actual}(t)). \tag{42}$$

This approach would for instance give the head's starting trajectory as shown in Figure 6.

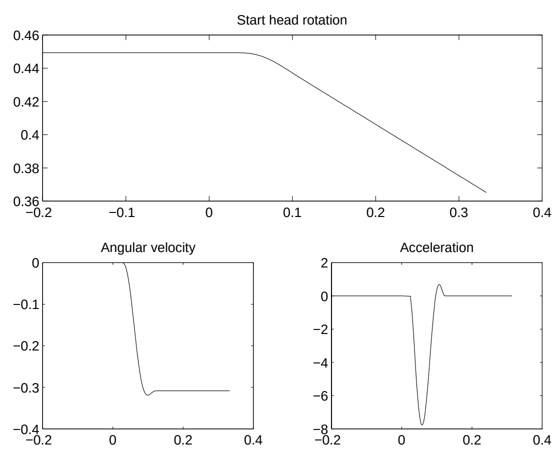


Figure 6: Head rotation when starting with zero velocity.

4 Optimal Control

Now we have found control laws for controlling the combined rotational movements, and we suspect that these controls yield reasonably good solutions. The method we used for determining how efficient any given solution was, was by calculating the energy produced in the movement. The energy for the different trajectories we have studied was calculated by a simple approximation:

$$E = \int_{t_0}^{t_f} |\ddot{\theta}\dot{\theta}| I \, dt \approx I \sum_{i=1}^{n} \Delta t_i |\ddot{\theta}(t_i)\dot{\theta}(t_i)|, \tag{43}$$

for a given set of n points on the interval $[t_0, t_f]$, with $t_0 \leq t_1 < t_2 < \ldots < t_{n-1} < t_n \leq t_f$. This approximative method gave the following results:

$$\begin{aligned} \psi(t) &= \psi_0 - (t - t_0) \Rightarrow E \approx 2.36 \cdot 10^{-3} \text{ J} \\ \psi(t) &= \psi_0 - (t - t_0)^2 \Rightarrow E \approx 6.42 \cdot 10^{-2} \text{ J} \\ \psi(t) &= \psi_0 - (t - t_0)^3 \Rightarrow E \approx 8.85 \cdot 10^{-2} \text{ J} \\ \psi(t) &= 0.5\phi_{\max} \sin 5(t - t_0) \Rightarrow E \approx 0 \text{ J} \\ \psi(t) &= 1.2\phi_{\max} \sin 5(t - t_0) \Rightarrow E \approx 3.52 \text{ J}. \end{aligned} \tag{44}$$

We now need something to compare these values with in order to find out if our solutions can be regarded as satisfactory.

Since we suspect that $\ddot{\theta}$ will be small at all times, we choose to investigate the minimization problem

$$\min \int_{t_0}^{t_f} \ddot{\theta}^2 \, dt \tag{45}$$

instead of

$$\min \int_{t_0}^{t_f} |\ddot{\theta}\dot{\theta}| \, dt \tag{46}$$

for the sake of simplicity, since the quadratic problem is likely to be less complicated, and hope that we still get a reasonably good solution anyway. The reason for this approach is that we are not really interested in the solution more than as a mere comparison to our previous solutions, in order to get a feeling for how good they are. If we start by letting

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \\ x_3 = \phi \\ x_4 = \dot{\phi} \\ x_5 = \psi \\ x_6 = \dot{\psi}, \end{cases} \tag{47}$$

the minimization problem becomes

$$\min \int_{t_0}^{t_f} (\text{sign}(x_1) f(|x_1), \text{sign}(x_1) x_2) + u(x_1) v)^2 \, dt, \tag{48}$$

under the dynamics

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mathrm{sign}(x_1) f(|x_1|, \mathrm{sign}(x_1) x_2) + u(x_1) v \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = F(x_1, x_2, \dot{x}_2, x_5, x_6, \dot{x}_6) \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = -2 \\ x_1(t_0) = \theta_0, \quad x_2(t_0) = 0, \quad x_3(t_0) = \phi_0, \\ x_4(t_0) = 0, \quad x_5(t_0) = \psi_0, \quad x_6(t_0) = 0, \\ x_1(t_f) = \theta_f, \quad x_2(t_f) = 0, \quad x_3(t_f) = \phi_f, \\ x_4(t_f) = 0, \quad x_5(t_f) = \psi_0 - (t_f - t_0)^2, \quad x_6(t_f) = -2(t_f - t_0), \end{cases} \tag{49}$$

if we let

$$\psi(t) = \psi_0 - (t - t_0)^2. \tag{50}$$

This system of differential equations gives the following Hamiltonian function:

$$\begin{aligned} H(x, y, v) \quad = \quad & (\mathrm{sign}(x_1) f(|x_1|, \mathrm{sign}(x_1) x_2) + u(x_1) v)^2 + y_1 x_2 \\ + \quad & y_2 (\mathrm{sign}(x_1) f(|x_1|, \mathrm{sign}(x_1) x_2) + u(x_1) v) + y_3 x_4 \\ + \quad & y_4 F(x_1, x_2, \dot{x}_2, x_5, x_6, \dot{x}_6) + y_5 x_6 - 2 y_6, \end{aligned} \tag{51}$$

where y is the dual state system variable along the optimal trajectory $\hat{x}$. According to Pontryagin's Maximum Principle (PMP) [1], we have

$$\dot{y}_j = -\frac{\partial H(\hat{x}, y, \hat{v})}{\partial x_j}, \tag{52}$$

which gives the dual system as

$$\begin{cases} \dot{y}_1 = (\mathrm{sign}(\hat{x}_1) \frac{\partial f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2)}{\partial x_1} + \frac{du(\hat{x}_1)}{x_1} \hat{v}) \\ \quad \times (-2(\mathrm{sign}(\hat{x}_1) f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2) \\ \quad + u(\hat{x}_1)\hat{v}) - y_2 - y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial \dot{x}_2}) - y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial x_1} \\ \dot{y}_2 = \mathrm{sign}(\hat{x}_1) \frac{\partial f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2)}{\partial x_2} (-2(\mathrm{sign}(\hat{x}_1) f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2) \\ \quad + u(\hat{x}_1)\hat{v}) - y_2 - y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial \dot{x}_2}) - y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial x_2} - y_1 \\ \dot{y}_3 = 0 \\ \dot{y}_4 = -y_3 \\ \dot{y}_5 = -y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial x_5} \\ \dot{y}_6 = -y_4 \frac{\partial F(\hat{x}_1, \hat{x}_2, \dot{\hat{x}}_2, \hat{x}_5, \hat{x}_6, \dot{\hat{x}}_6)}{\partial x_6} - y_5. \end{cases} \tag{53}$$

According to PMP, the optimal control law is given by

$$H(\hat{x}, y, \hat{v}) = \min_{v \in \mathcal{U}} H(\hat{x}, y, v), \tag{54}$$

where $\mathcal{U}$ is the set of allowed control laws.

This boundary value problem has to be solved numerically, and if we use the shooting method [4], [18] on this problem, we are given

$$E \approx 1.75 \cdot 10^{-2} \text{ J}, \tag{55}$$

which is more than a 70% improvement compared to our previous result. However, it must be stressed that this optimal control strategy only works when the tracked object's trajectory is completely known. This approach could however represent the case when the same tracking task is repeated, and we therefore have complete knowledge of the tracked object's trajectory. This could for instance be the case when we are reading and basically performing the same tracking task over and over again.

5 Conclusions

When it comes to the developed model, the weakest part is probably that of trying to model muscles as second order springs, since an actual muscle has a dynamics that is more complicated than that. However, this approach has the major advantage that it makes the mathematics reasonably simple. It is also sufficiently complete when it comes to actually start thinking about how to control the head and the eye muscles simultaneously. As we have seen, this is a non-trivial problem.

The control strategy we chose to use was based on a desire to keep the energy produced in the movement small, since we believed this to be a physically reasonable approach. We therefore let the angular acceleration of the head be zero most of the time, since this would make the energy small.

When minimizing the square of the angular acceleration of the head, the energy in both the piecewise linear case and in the optimal case, were of the same power of ten, which must be regarded as acceptable. However, it must be stressed that our control strategy only works when the tracked object's trajectory is known, since our controls are, among other things, based on a knowledge of the total time that tracking is to be conducted.

When it comes to physical adequacy, it can be worth comparing our results to the trajectories found in Guitton's *Eye-Head Coordination in Gaze Control*. It turns out that our piecewise linear approach is not so bad after all, since an actual combined movements seem to have somewhat of the same piecewise linear characteristics as our trajectories, even though they are somewhat more complex. This should not, however, disqualify our model as not being an interesting step towards an understanding of the complex behavior of human head-eye-coordination.

6 Simulations

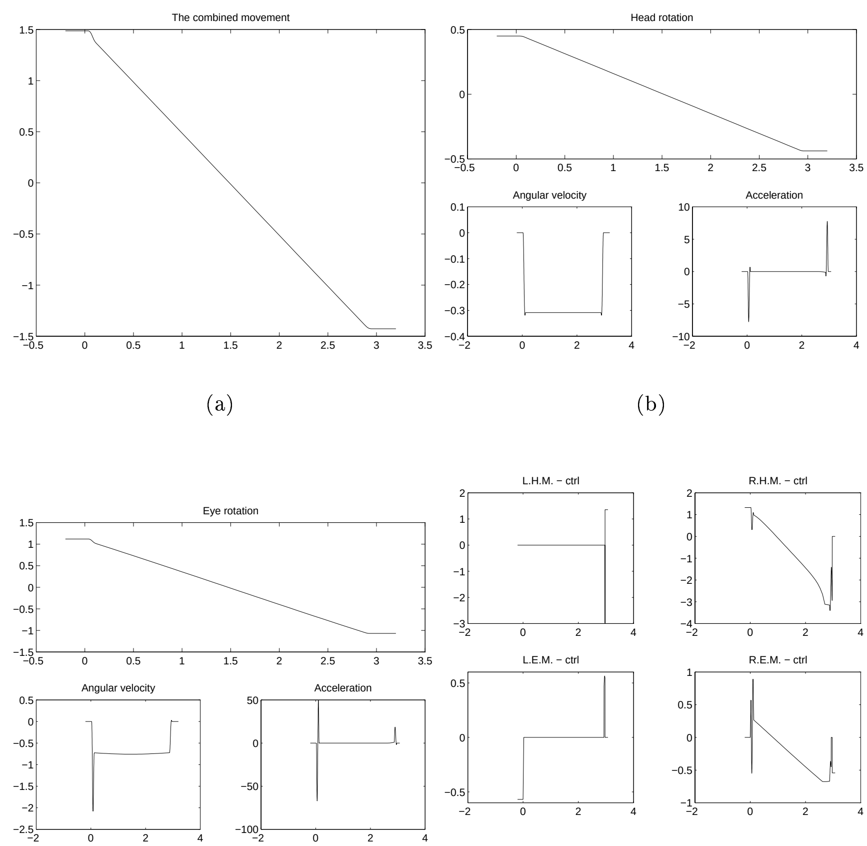
In this appendix, the different graphs produced when simulating the movements are presented. The controls are presented for each individual muscle, with the notation

L.H.M - ctrl	:	Left Head Muscle Control
R.H.M - ctrl	:	Right Head Muscle Control
L.E.M - ctrl	:	Left Eye Muscle Control
R.E.M - ctrl	:	Right Eye Muscle Control.

It can be worth noticing that each pair of muscles have the desired property that

$$v_1(t)v_2(t) = 0 \quad \forall t.$$

In each section, the energy produced by the head rotation is also listed, calculated by equation 43.



(a) (b) (c) (d)

Figure A-1: Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)$. No corrections from the zero head acceleration was necessary. The energy produced in the movement was $E \approx 2.36 \cdot 10^{-3}$ J.

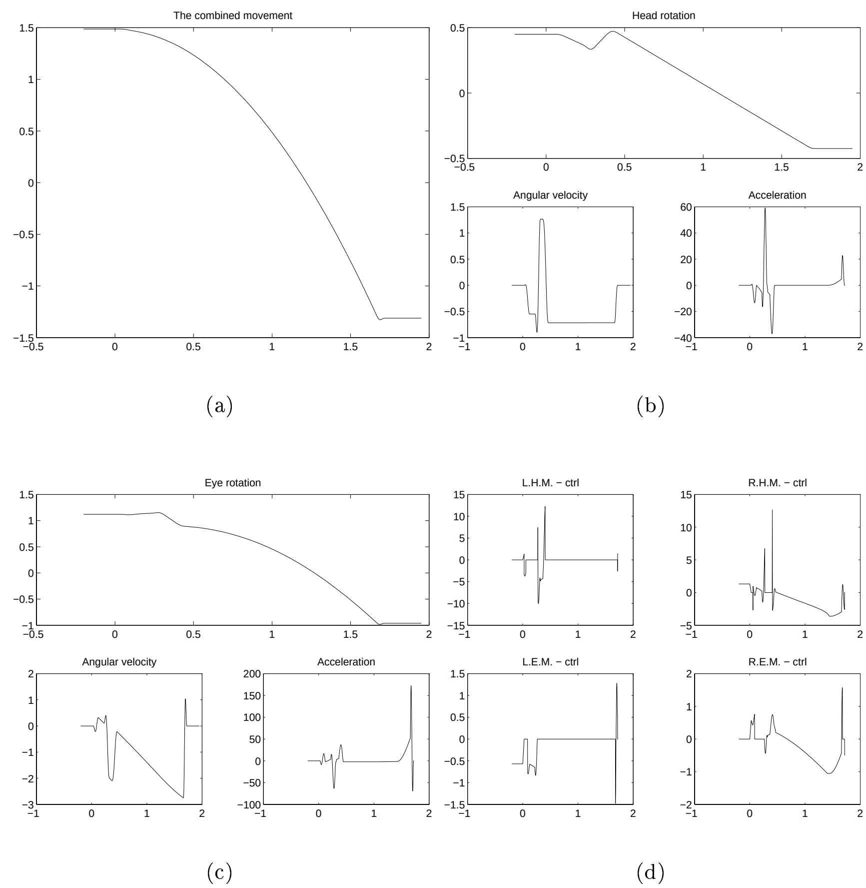


Figure A-2: Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)^2$. One correction of the head acceleration was necessary, since the eye was rotating out of bound. The energy produced was $E \approx 6.42 \cdot 10^{-2}$ J.

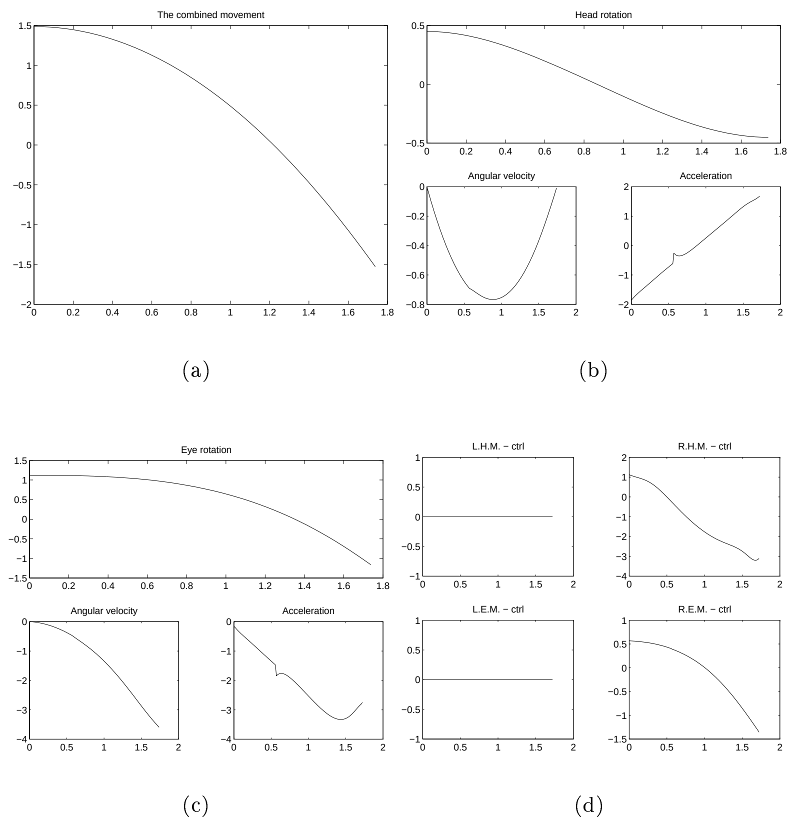


Figure A-3: Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)^2$, and when the controls were designed to minimize the square of the angular acceleration of the head, using Pontryagin's Maximum Principle. The energy produced was $E \approx 1.75 \cdot 10^{-2}$ J in this optimal case. This can be compared to our previous $6.42 \cdot 10^{-2}$ J, but in this case, we have no guarantee that the trajectories are three times continuously derivable.

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