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The Minimum Time Function with Unbounded Controls*

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Abstract

The classical time–optimal problem is investigated with the sole hypothesis on the controls that they are bounded in the L^1 norm. In fact, we allow the controls to be unbounded in the L^∞ norm and we do not assume any of the growth conditions that prevent the exploitation of larger and larger controls. An assumption of *controllability with zero energy* (which means that only the bounded component of the control is used to reach the target from a neighborhood of it) is proved to be sufficient for the continuity of the time–optimal map. Under the same assumption this map turns out to be the unique solution of a suitable Bellman equation with boundary conditions of mixed type. The result relies essentially on a reparameterization technique, which, in particular, allows one to replace the (discontinuous) conventional Hamiltonian with a more regular one.

Key words: time optimal problem, unbounded controls, dynamic programming

AMS Subject Classifications: 49L20, 49N25, 93C10

1 Introduction

The minimum time problem has commonly been studied under the hypothesis that the set of available velocities were bounded (see e.g. [1, 2, 3, 4, 5, 6], [9, 10], [12], [17, 18], [51], [53, 55], [57] and the references therein). Alternatively, a growth condition has been assumed to make the use of large controls disadvantageous. In fact, many applications show these assumptions to be quite reasonable, and both the boundedness of the controls and the growth hypotheses imply that the problem has important regularity properties. For example, the corresponding Hamiltonian turns out to be continuous.

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However, the mathematical models describing a certain class of applications lack both boundedness and growth. Examples of this can be found in rational mechanics when holonomic constraints are used as controls (see [15],[19], [43]). Further examples come from space navigation theory (see [28], [31]) and from advertising modelling (see [23]). Indeed, these applications share the fact that the velocity field and the Lagrangian depend linearly on the derivative of a certain control parameter. When this derivative is itself regarded as a control there are often no reasons to justify a boundedness assumption on it. Moreover, the linear (or sublinear) dependence of both the dynamics and the Lagrangian rules out the growth hypothesis as well. This situation represents the control-theoretic analogue of what happens in calculus of variations with slow growth (see e.g [20]). For this reason we refer to these problems as to slow growth control problems.

Slow growth control problems have been investigated since the early sixties (see e.g. [41], [44], [46]). In the first approaches to the problem the unbounded controls appeared linearly in both the dynamics and the Lagrangian, and their coefficient did not depend on the state variable. Such an hypothesis allowed for a measure theoretical interpretation of both the equations of motion and the integral cost. In particular it was possible to give a robust notion of solution corresponding to a control represented by a measure.

More recently genuine nonlinear slow growth problems have been addressed (see e.g. [7], [11], [13], [14], [16], [22, 23, 24], [29, 30], [32, 33, 34, 35, 36, 37, 38, 39, 40], [44, 45], [47, 48, 49, 50]). We mean, for instance, the case where the dynamics and the Lagrangian depend linearly on the unbounded controls but the coefficients of the latter are functions of the state as well. More generally one considers the case where the fields and the Lagrangian depend sublinearly on the unbounded controls. It is known (see [13]) that in these cases a measure theoretical interpretation of the dynamic equations leads to an ill–posed problem, for no definition of solution exists having continuous dependence on a control which degenerates into a measure. Actually a different approach, based on embedding into space–time, suits the problem better. Here, according to this line of investigation, we address the minimum time problem with unbounded controls.

More precisely we consider the control system

$$\dot{x} = f(t, x, v, \xi) \qquad x(\bar{t}) = \bar{x} \tag{1.1}$$

where $(t, x) \in \boldsymbol{R}^{1+n}$, v is a standard—i.e., bounded— control, and ξ belongs to a closed cone $C \subset \boldsymbol{R}^m$. Given a closed target $\mathcal{T} \subset \boldsymbol{R}^n$, the goal of the problem consists in choosing a control policy (v, ξ) so that the corresponding trajectory reaches the target $\mathcal{T}$ in a time $t_f - \bar{t}$ as short as possible. Though we allow the control ξ to have unbounded L^∞ norm, we impose an L^1 bound on it. Namely we assume that

$$\int_{\bar{t}}^{t_f} |\xi(t)| dt \leq K - \bar{k} \tag{1.2}$$

where $K > 0$ and $\bar{k} \in [0, K]$. We can think of $\bar{k}$ as an initial condition which prescribes the maximal amount $K - \bar{k}$ of *energy* available to the control ξ.

Besides some regularity hypotheses we assume the following slow growth condition on the field f: there exists a map $f^\infty = f^\infty(t, x, v, w)$, called the *recession map of* f, with the same regularity as f, such that

$$f^\infty(t, x, v, w) = \lim_{r \to 0} f(t, x, v, \frac{w}{r})r.$$

The existence of the recession map f^∞ implies that f grows at most linearly in ξ as $|\xi|$ tends to infinity.

An obvious example of such an f is given by

$$f_1(t, x, v, \xi) \doteq g_0(t, x, v) + \sum_{i=1}^{m} g_i(t, x, v)\xi_i,$$

where $g_0, g_1, \ldots, g_m$ are vector fields with standard regularity properties. Setting

$$\bar{f}(t, x, v, w_0, w) \doteq \begin{cases} f(t, x, v, \frac{w}{w_0})w_0 & \text{if} \quad w_0 \neq 0 \\ f^\infty(t, x, v, w) & \text{if} \quad w_0 = 0 \end{cases}$$

we obtain

$$\bar{f}_1(t, x, v, w_0, w) \doteq g_0(t, x, v)w_0 + \sum_{i=1}^{m} g_i(t, x, v)w_i.$$

Further examples can be found in [38]. We remark that, due to the slow growth assumption, it may happen that minimizing sequences of measurable controls (v_n, ξ_n) do not converge to a measurable control (v, ξ) (note, incidentally, that this phenomenon is *not* related to the lack of convexity: it may occur even if convexity assumptions hold). For example ξ_n could converge to a distribution, say a delta function. As already remarked, in that case the attempt to give a distributional sense to equation (1.1) fails because of the nonlinear nature of the problem. Consider for example a dynamics like f_1 above: if the vector fields $g_1, \ldots, g_m$ are independent of x and v, it can be shown that a distributional approach still works (see [46], [8]). As soon as the g_i's depend on (v, x), however, an obvious drawback arises when one tries to define a trajectory corresponding to a control (v, ξ) whose second component ξ is a delta function.

These kinds of problems have been thoroughly investigated in [38] and [35] by embedding the system (1.1) in the following space–time differential system:

$$\begin{array}{l} \frac{dt}{ds} = w_0 \\ \frac{dx}{ds} = \bar{f}(t,x,v,w_0,w) \\ (t,x)(0) = (\bar{t},\bar{x}), \end{array} \tag{1.3}$$

where s is a pseudo–time parameter with respect to which the time t is a non-decreasing, Lipschitz continuous function. It is remarkable that as soon as $w_0 = 0$ on an interval $[s_1, s_2]$, the time variable remains equal to a constant $\tilde{t}$ on this interval, while the space variable x evolves *according to the dynamics* $\bar{f}(\tilde{t}, x, v, 0, w) = f^\infty(\tilde{t}, x, v, w)$. In other words the *jump* of x at $\tilde{t}$ is determined by the values of the control w on the whole interval $[s_1, s_2]$ (while in the case where the distributional approach can be applied, this jump depends only on the integral of w over $[s_1, s_2]$).

In the space–time formulation of the problem the integral constraint (1.2) has to be replaced by the inequality

$$\int_0^1 |w(s)|ds \le K - \bar{k}. \tag{1.4}$$

It can be shown that each space-time trajectory $(t,x)(s)$ can be approximated by (reparametrizations of) the graphs $(t, x_n(t))$ of trajectories $x_n(t)$ of the original system (1.1). However, on the one hand it is *not* true that each space–time trajectory reaching the target $\mathcal{T}$ can be approximated by (the graphs of) trajectories of (1.1) that reach $\mathcal{T}$. On the other hand such an approximability property is essential if one wishes to consider (1.3) as an *extension* of (1.1) (see [56]). In fact, we shall assume the following hypothesis: **Hypothesis (H)**. *For every $R > 0$ there exist $\nu_R > 0$ and $\sigma_R > 0$ such that for every $(t,x) \in (\boldsymbol{R} \times (\mathcal{T}_{\sigma_\mathcal{R}} \setminus \mathcal{T})) \cap \mathcal{B}_{\infty+\backslash}(\prime, \mathcal{R})$ there exists $v_{t,x} \in V$ such that*

$$f(t,x,v_{t,x},0) \cdot \frac{x - \pi(x)}{|x - \pi(x)|} \le -\nu_R$$

for some $\pi(x) \in \mathcal{T}$ such that $|x - \pi(x)| = d_\mathcal{T}(x)$, where $d_\mathcal{T}(x)$ denotes the distance between $\mathcal{T}$ and the point $x \in \boldsymbol{R}^n$ and for each $\rho > 0$ $\mathcal{T}_\rho$ denotes the open set $\{x \in \boldsymbol{R}^n : d_\mathcal{T}(x) < \rho\}$.

Hypothesis (H) can be thought of as an assumption of *controllability with zero energy.* This means nothing but a version of the classical inwards pointing field condition for the vectogram $f(t,x,v,0)$ at the points $x \in \partial\mathcal{T}$.

Under Hypothesis (H) and for a given amount $K - \bar{k}$ of maximal available energy we prove the following: if a space–time trajectory of (1.3) starting from a point $(\bar{t}, \bar{x})$ reaches a point $y \in \mathcal{T}$, then a point $z \in \mathcal{T}$ near y can be reached by a trajectory of the original system (1.1) starting from $\bar{x}$ at $\bar{t}$.

Let us denote by $T(\bar{t}, \bar{x}, \bar{k})$ the infimum of the times one needs to reach the target starting from $\bar{x}$ at time $\bar{t}$ with energy less than or equal to $K - \bar{k}$. Let $T_e(\bar{t}, \bar{x}, \bar{k})$ be the analogous quantity —i.e. the infimum of $\int_0^1 |w_0(s)|ds$— for the extended system (1.3). Under Hypothesis (H) we prove that $T = T_e$. This fact together with the above approximability argument makes the extended minimum time problem a proper extension of the original one.

The advantage of addressing T_e instead of T relies on the fact that T_e involves only *bounded controls.* This is the consequence of a suitable combination of three facts: first, the extended system (1.3) is invariant with respect to changes of the parameter s; secondly, each control must satisfy the integral constraint (1.4); finally, under Hypothesis (H) the map $T_e(= T)$ is locally bounded.

When Hypothesis (H) is in force, we prove (see Section 4) that T_e is continuous. Incidentally, this also provides an extension of the results concerning the case with bounded controls, in that our field f depends on the time variable as well. Actually, if f is Lipschitz continuous in (t, x, ξ), T_e turns out to be Lipschitz continuous. Let us note that, unlike the optimal time map for problems with bounded controls, T_e may happen to be equal to zero even at points not belonging to the target $\mathcal{T}$. In Section 5 we establish a Bellman equation for T_e which, unlike the formal Bellman equation, involves a continuous Hamiltonian. The boundary conditions satisfied by T_e on $0 \leq k < K$ are of Dirichlet type, while, for $k = K$, T_e is a supersolution of the established Bellman equation. This latter fact is not surprising in view of the results on constrained control problems (see e.g. [52], [39]): actually, the integral constraint (1.4) can be interpreted as a state constraint for the variable $k(s) = \bar{k} + \int_0^s |w(\sigma)|d\sigma$.

By means of a suitable *Kruskov-type* transformation of the dependent variable we prove that T_e is the unique (viscosity) solution of the established boundary value problem. Two appendices, where some technical results are proved, conclude the paper.

Notation: Throughout the paper we denote by $B_m[x_0, r]$ the closed ball of $\boldsymbol{R}^m$ with center in x_0 and radius r and by $B_m(x_0, r)$ its interior. We denote by $\| \cdot \|_\infty$ the sup norm. Given the set A we indicate by ∂A its boundary and by *Int* A its interior. For a closed set $\mathcal{T} \subseteq \boldsymbol{R}^n$, $d_{\mathcal{T}}(x)$ denotes the distance between $\mathcal{T}$ and the point $x \in \boldsymbol{R}^n$ and $\mathcal{T}_\rho$ denotes the open set $\{x \in \boldsymbol{R}^n : d_{\mathcal{T}}(x) < \rho\}$. We call *modulus* each positive, continuous, nondecreasing function from $\boldsymbol{R}$ to $\boldsymbol{R}$ which maps zer–t–zero. Finally by *cone* we mean a subset of a vector space closed under multiplication by non negative scalars.

2 The Control System and the Minimum Time Function

We consider a control system of the form

$$\begin{cases} \dot{x} = f(t,x,v,\xi) \\ x(\bar{t}) = \bar{x}, \end{cases} \tag{2.1}$$

where v is a *conventional* control which takes values in a compact set $V \subset \boldsymbol{R}^q$, while the control ξ is unbounded and takes values in a closed cone $C \subset \boldsymbol{R}^m$.

We assume the following hypotheses on the vector field f:

i) $f \in \mathcal{C}(\boldsymbol{R}^{1+n} \times V \times C, \boldsymbol{R}^n)$ and for every compact subset $Q \subset \boldsymbol{R}^{1+n}$ there exists a positive constant $L = L_Q$ and a modulus $\omega = \omega_Q$ satisfying

$$|f(t_1,x_1,v,\xi) - f(t_2,x_2,v,\xi)| \leq (1+|\xi|)(L|x_1 - x_2| + \omega(|t_1 - t_2|)), \tag{2.2}$$

for all $(t_1,x_1,v,\xi),(t_2,x_2,v,\xi) \in Q \times V \times C$;

ii) there exists a continuous nondecreasing function $M(t) > 0$ such that

$$|f(t,x,v,\xi)| \leq M(t)(1+|\xi|)(1+|x|) \tag{2.3}$$

for every $(t,x,v,\xi) \in \boldsymbol{R}^{1+n} \times V \times C$;

iii) (**slow growth**) there exists a map $f^\infty \in \mathcal{C}(\boldsymbol{R}^{1+n} \times V \times C,\ \boldsymbol{R}^n)$, called the *recession function of* f, such that

$$\lim_{r \to +\infty} r^{-1} f(t,x,v,r\xi) = f^\infty(t,x,v,\xi) \tag{2.4}$$

uniformly on compact sets of $\boldsymbol{R}^{1+n} \times V \times C$.

We now introduce the set of controls

$$W(\bar{t}) \doteq \{(v,\xi) \in \bigcup_{T > \bar{t}} \mathcal{B}([\Box, \mathcal{T}], \mathcal{V} \times \mathcal{C})\},$$

where $\mathcal{B}([a,b],E)$ denotes the set of Borel measurable functions from $[a,b]$ into a metric space E which are Lebesgue integrable. Let $K > 0$ be fixed, and for every $\bar{k} \in [0,K]$, let us set

$$W(\bar{t},\bar{k}) \doteq \{(v,\xi) \in W(\bar{t}) : \int_{\bar{t}}^{T} |\xi(s)|\, ds \leq K - \bar{k}\},$$

where, for every pair (v,ξ), the involved domain of integration $[\bar{t},T]$ coincides with the domain of definition of (v,ξ).

The assumptions (2.2) and (2.3) ensure existence, uniqueness and exponential growth of the solution to (2.1) for every $(v,\xi)\in W(\bar{t})$. Such a solution will be indicated by $x_{\bar{t},\bar{x}}(v,\xi)(t)$.

Let $\mathcal{T}$ be a closed subset of $\boldsymbol{R}^n$. For every $(\bar{t},\bar{x})\in\boldsymbol{R}^{1+n}$ and every $(v,\xi)\in W(\bar{t})$ define

$$\theta_{\bar{t},\bar{x}}(v,\xi)=\begin{cases}\inf\{t-\bar{t}\}\,, & \text{if } x_{\bar{t},\bar{x}}(v,\xi)(t)\in\mathcal{T} \text{ for some } t\geq\bar{t}\\ +\infty, & \text{if } x_{\bar{t},\bar{x}}(v,\xi)(t)\notin\mathcal{T} \text{ for any } t\geq\bar{t}.\end{cases}$$

A control will be called *admissible* for $(\bar{t},\bar{x})$ if $\theta_{\bar{t},\bar{x}}(v,\xi)<+\infty$, and a trajectory corresponding to an admissible control will be called *admissible.* The minimum time function is then defined as

$$T(\bar{t},\bar{x},\bar{k})\doteq\inf_{(v,\xi)\in W(\bar{t},\bar{k})}\left\{\theta_{\bar{t},\bar{x}}(v,\xi)\right\}.$$

A problem arises immediately. As it will be clear from Example 3.1 below, the lack of any L^∞-bound on the control ξ and the slow growth assumption possibly yield minimizing sequences of trajectories whose derivative are larger and larger in the L^∞ norm. In other words these trajectories *tend* to a discontinuous map. To tackle this problem we follow the approach already exploited e.g. in [32, 33, 34, 35, 36, 37, 38, 39, 40], that is, we *embed* the dynamics of (2.1) into a new dynamics in which the variable t is treated as a space variable which is *nondecreasing* with respect to a new parameter s. In this extended setting, as soon as the function T is bounded, the minimizing trajectories are uniformly Lipschitz continuous provided that the parameter s is suitably chosen (see Proposition 2.3).

For the sake of completeness we list here some definitions and results from [40] which describe this embedding.

Definition 2.1. *A space-time control is a triple belonging to the set* Γ *defined by*

$$\Gamma\doteq\mathcal{B}([\prime,\infty],\mathcal{V}\times[\prime,+\infty)\times\mathcal{C}).$$

Moreover we set

$$\Gamma(k)\doteq\left\{(v,w_0,w)\in\Gamma:\int_0^1|w(s)|\,ds\leq K-k\right\}.$$

Definition 2.2. *For every* $(t,x)\in\boldsymbol{R}^{1+n}$ *and every triple* $(v,w_0,w)\in\Gamma$ *we set*

$$\overline{f}(t,x,v,w_0,w)\doteq\begin{cases} f\left(t,x,v,\frac{w}{w_0}\right)\cdot w_0 & \textit{if }\ w_0\neq 0\\ f^\infty(t,x,v,w) & \textit{if }\ w_0=0.\end{cases}$$

The function $\overline{f}$ is therefore the continuous extension of the map $f\left(t, x, v, \frac{w}{w_0}\right) w_0$ to the set $\mathcal{D}_\rceil \doteq \boldsymbol{R}^{1+n} \times V \times [0, +\infty) \times C$.

We consider now the new, extended system, also called *space-time* system,

$$\begin{cases} t'(s) = w_0(s) \\ x'(s) = \overline{f}(t(s), x(s), v(s), w_0(s), w(s)) \\ (t(0), x(0)) = (\bar{t}, \bar{x}), \end{cases} \tag{2.5}$$

where differentiation is done with respect to the new parameter $s \in [0, 1]$. A solution of this system will be indicated by either $(\bar{t} + \int_0^s w_0(\sigma)\, d\sigma,$ $x_{\bar{t},\bar{x}}(v, w_0, w)(s))$ or $(t, x)_{\bar{t},\bar{x}}(v, w_0, w)(s)$.

Let us define the sets

$$\Gamma^+ \doteq \left\{ (v, w_0, w) \in \Gamma \quad \text{such that :} \quad w_0(s) > 0 \quad \text{for a.e.} \quad s \in [0, 1] \right\}$$

and

$$\Gamma^+(k) \doteq \Gamma^+ \cap \Gamma(k).$$

Proposition 2.1. *If $(v, \xi) \in \mathcal{B}([\bar{\sqcup}, \mathcal{T}], \mathcal{V} \times \mathcal{C})$, let us consider $s : [0, 1] \to ([\bar{t}, T] \times C)$ such that $s \mapsto (t(s), u(s))$ is any Lipschitz continuous parameterization of the graph of $t \mapsto u(t) \doteq \int_{\bar{t}}^t \xi(\tau) d\tau$ with $t'(s) > 0$. Then, setting $w_0(s) \doteq t'(s)$, $w(s) \doteq u'(s)$ and $v(s) \doteq v \circ t(s)$, one has that $x(t)$ is the solution to (2.1) corresponding to (v, ξ) if and only if $(t(s), x \circ t(s))$ is the solution to (2.5) corresponding to the control $(v(s), w_0(s), w(s))$, the latter belonging to Γ^+. Moreover, let $(v, w_0, w) \in \Gamma^+$ and let $(t, x)_{\bar{t},\bar{x}}(s)$ denote the corresponding solution to (2.5). Then the position*

$$(\tilde{v}, \xi)(t) \doteq (v, \frac{w}{w_0}) \circ s(t),$$

where $s(\cdot)$ is the inverse of $t(\cdot)$, defines almost everywhere in $[\bar{t}, T], T = t(1)$, a control belonging to $W(\bar{t})$ and the solution $\tilde{x}$ to (2.1) corresponding to $(\tilde{v}, \xi)$ verifies

$$\tilde{x} \circ t(s) = x(s)$$

for every $s \in [0, 1]$.

Proof: The first part of the Proposition follows from the uniqueness of the solution to (2.1) and (2.5). The same uniqueness property implies the second part, provided $(\tilde{v}, \xi)(\cdot)$ belongs to $W(\bar{t})$. To prove this latter fact, set

$$\phi(s) \doteq \int_0^s w(\sigma)\, d\sigma, \quad u(t) = \phi \circ s(t).$$

Since $(w_0, w) \in \Gamma^+$, there exists a null subset $\mathcal{N} \subset [\prime, \infty]$ such that $t(\cdot)$ is differentiable on $[0,1] \setminus \mathcal{N}$ and

$$t'(s) = w_0(s) > 0 \quad \forall s \in [0,1] \setminus \mathcal{N}.$$

Moreover $s(\cdot)$ is absolutely continuous (see e.g. [20]). In particular there exists a null subset $\mathcal{M} \subset [\bar{\square}, \mathcal{T}]$ such that $s(\cdot)$ is differentiable on $[\bar{t}, T] \setminus \mathcal{M}$. Let us set $\mathcal{O} \doteq \mathcal{M} \cup \sqcup(\mathcal{N})$ and let us observe that $\mathcal{O}$ is a null set, for $t(\cdot)$ is absolutely continuous. Hence $u(\cdot)$ is differentiable almost everywhere, namely on $[\bar{t}, T] \setminus \mathcal{O}$, and one has

$$u'(t) = \frac{w}{w_0} \circ s(t) = \xi(t) \quad \forall t \in [\bar{t}, T] \setminus \mathcal{O}.$$

Since $u(\cdot)$ is absolutely continuous, it follows that (any extension to $[\bar{t}, T]$ of) $\xi(\cdot)$ is integrable. Since the L^1- equivalence class of $(\tilde{v}, \xi)(\cdot)$ contains a Borel map, the Proposition is proved. □

Notice that a Lipschitz continuous parameterization $(t(s), u(s))$ as in the previous statement always exists. Indeed it is sufficient to consider the inverse $t(s)$ of the map $s(t) \doteq \frac{\int_{\bar{t}}^{t} |(1,\xi(s'))|}{\int_{\bar{t}}^{T} |(1,\xi(s'))|} ds'$ and set $u(s) \doteq u \circ t(s)$.

In the sense specified by the above proposition the sets $\bigcup_{\bar{t} \in \boldsymbol{R}} W(\bar{t})$ and $\bigcup_{\bar{t} \in \boldsymbol{R}} W(\bar{t}, \bar{k})$ can be identified with Γ^+ and $\Gamma^+(\bar{k})$, respectively, for every $\bar{k} \in [0, K]$. For this reason the trajectories corresponding to Γ^+ will be called *regular*. Let $\sigma : [0,1] \to [0,1]$ be an increasing, surjective map, continuous with its inverse, and $(v, w_0, w) \in \Gamma$. The map $(\hat{v}, \hat{w}_0, \hat{w}) \doteq (v \circ \sigma, (w_0 \circ \sigma)\sigma', (w \circ \sigma)\sigma')$ defines almost everywhere a space-time control, as it is easy to check.

Proposition 2.2. [40] *We have*

$$x_{\bar{t},\bar{x}}(v, w_0, w) \circ \sigma(s) = x_{\bar{t},\bar{x}}(\hat{v}, \hat{w}_0, \hat{w})(s)$$

for every $s \in [0,1]$.

We recall now from [38] the notion of canonical parameterization. Let $(v, w_0, w) \in \Gamma$ and let σ_c be the map from [0,1] into itself defined by

$$\sigma_c(s) \doteq \frac{\int_0^s |(w_0, w)|(s) ds}{\int_0^1 |(w_0, w)|(s) ds}.$$

If $(w_0, w) = 0$ in [0,1] we set $(v^c, w_0^c, w^c) \doteq (v, w_0, w)$ otherwise we set

$$(v^c \circ \sigma_c, w_0^c \circ \sigma_c \cdot \frac{d\sigma_c}{ds}, w^c \circ \sigma_c \cdot \frac{d\sigma_c}{ds}) \doteq (v(s), w_0(s), w(s)). \qquad (2.6)$$

In [36] the following Proposition was proved:

Proposition 2.3. *The relation (2.6) defines a measurable map* (w_0^c, w^c) *on* $[0,1]$ *so that* $|w_0^c, w^c|(s) = \int_0^1 |(w_0, w)|(s)ds$ *a.e. in* $[0,1]$. *Moreover (2.6) defines a univalued Borel measurable map* v^c *almost everywhere in* $[0,1]$. *Finally the equality*

$$(t,x)_{\bar{t},\bar{x}}(v,w_0,w)(\sigma_c^{-1}(\{s\})) = (t,x)_{\bar{t},\bar{x}}(v^c, w_0^c, w^c)(s)$$

holds for every $s \in [0,1]$.

The triple (v^c, w_0^c, w^c) is called *canonical parameterization* of (v, w_0, w). In the next lemma we state some properties that $\overline{f}$ inherits from f:

Lemma 2.1. *The function* $\overline{f}$ *is continuous in* $\mathcal{D}_\rceil$. *Moreover assumptions (2.2) and (2.3) on* f *imply that* $\overline{f}$ *satisfies:*

i) *for every compact subset* $Q \subset \boldsymbol{R}^{1+n}$ *there exist* $L = L_Q$ *and* $\omega = \omega_Q$ *such that*
$|\overline{f}(t_1,x_1,v,w_0,w) - \overline{f}(t_2,x_2,v,w_0,w)| \leq$
$(w_0 + |w|)(L|x_1 - x_2| + \omega(|t_1 - t_2|)),$
$\forall (t_1,x_1,v,w_0,w),(t_2,x_2,v,w_0,w) \in Q \times V \times [0,+\infty) \times C$;

ii) $|\overline{f}(t,x,v,w_0,w)| \leq M(t)(w_0+|w|)(1+|x|),\ \forall (t,x,v,w_0,w) \in \mathcal{D}_\rceil$, *where* $M(t)$ *is the same as in (2.3);*

iii) $\overline{f}(t,x,v,\alpha w_0,\alpha w) = \alpha \overline{f}(t,x,v,w_0,w),\ \forall (t,x,v,w_0,w) \in \mathcal{D}_\rceil$, $\forall \alpha \in \boldsymbol{R}$.

Setting

$$M_{\bar{t},\bar{x}} = \max_{(v,w_0,w)\in V\times(B_{1+m}[0,1]\cap([0,+\infty)\times C))} |\overline{f}(\bar{t},\bar{x},v,w_0,w)| + \sqrt{2},$$

by ii) of the previous lemma we have that

$$|\overline{f}(\bar{t},\bar{x},v,w_0,w)| \leq M_{\bar{t},\bar{x}} \cdot |(w_0,w)|.$$

Moreover Gronwall's Lemma gives

$$|x_{\bar{t},\bar{x}}(v,w_0,w)(s)| \leq |\bar{x}| + e^{\int_0^s M(t(s'))\sqrt{2}|(w_0,w)(s')|\,ds'} - 1 \qquad (2.7)$$

where $t(s) = \bar{t} + \int_0^s w_0(s')\,ds'$.

Remark 2.1. It is clear that for $\bar{t}$ belonging to a bounded set and for all space-time controls (v,w_0,w) such that $\int_0^1 w_0(s)\,ds \leq S$, where S is a given positive constant, $M(\bar{t} + \int_0^1 w_0(s)\,ds)$ is bounded. Therefore, for these same controls, since $\int_0^1 |w(s)|\,ds \leq K$, and for $(\bar{t},\bar{x}) \in Q$, Q compact, there exists a compact Q' containing all the extended trajectories $(t,x)_{\bar{t},\bar{x}}(v,w_0,w)(s)$.

Proposition 2.4. *For every $(\bar{t},\bar{x}) \in \boldsymbol{R}^{1+n}$ the set of regular trajectories issuing from $(\bar{t},\bar{x})$ is dense in the set of space-time trajectories issuing from the same initial data.*

The previous proposition is based on the fact that given a control $(v,w_0,w) \in \Gamma(k)$, for every $n \in \boldsymbol{N}$ we define the new control (v,w_{0_n},w) by setting $w_{0_n} = \frac{1}{n} + w_0$. The controls (v,w_{0_n},w) belong to $\Gamma^+(k)$ and the corresponding trajectories approximate in the sup-norm the trajectory corresponding to (v,w_0,w).

Finally we give a result of approximability of trajectories. It is a slight modification of Proposition 3.1 of [39] and we will present its proof in Appendix 1 just for the sake of self-consistency.

Proposition 2.5. *Fix $\bar{y} = (\bar{t},\bar{x},\bar{k}) \in \boldsymbol{R}^{1+n} \times [0,K]$, $(\overline{v},\overline{w}_0,\overline{w}) \in \Gamma(\bar{k})$ and $\delta > 0$. Then there exists a modulus $\rho(\cdot)$ such that for every $y = (t,x,k) \in B_{2+n}(\bar{y},\delta)$ there exists a control $(v,w_0,w) \in \Gamma(k)$ with $\int_0^1 w_0 ds \leq \int_0^1 \overline{w}_0 ds + \delta$, $\int_0^1 |w| ds \leq \int_0^1 |\overline{w}| ds$ and*

$$\|x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w}) - x_{t,x}(v,w_0,w)\|_\infty \leq \rho(\delta). \tag{2.8}$$

Moreover for every compact $Q \subseteq \boldsymbol{R}^{1+n}$ and $S > 0$ one can choose $\rho(\cdot)$ independent of $(\bar{t},\bar{x},\bar{k})$ and $(\overline{v},\overline{w}_0,\overline{w})$, provided $(\bar{t},\bar{x}) \in Q$ and $\int_0^1 \overline{w}_0\, ds \leq S$.

Proof: See Appendix 1. □

If we assume Hypothesis (L) below, then estimate (2.8) can be improved. **Hypothesis (L)**For every compact subset $Q \subseteq \boldsymbol{R}^{1+n}$ there exists a constant $\bar{L}_Q$ such that

$$\begin{aligned}|f(t_1,x_1,v,\frac{w_1}{w_{0_1}})w_{0_1} - f(t_2,x_2,v,\frac{w_2}{w_{0_2}})w_{0_2}| \leq \\ \bar{L}_Q(|t_1-t_2| + |x_1-x_2| + |w_{0_1}-w_{0_2}| + |w_1-w_2|),\end{aligned}$$

for all $(t_1,x_1,v,w_{0_1},w_1),(t_2,x_2,v,w_{0_2},w_2) \in Q \times V \times (0,+\infty) \times C$.

For instance Hypothesis (L) is verified if f is affine with respect to ξ and locally Lipschitz continuous with respect to (t,x).

Corollary 2.1. *Assume hypothesis* (L). *Fix $Q \subseteq \boldsymbol{R}^{1+n}$ and $S > 0$. Then there exists a positive constant $\bar{\bar{L}}_Q$ such that for every $\bar{y} = (\bar{t},\bar{x},\bar{k})$ $y = (t,x,k)$ with $(\bar{t},\bar{x}),(t,x) \in Q$ and for every $(\overline{v},\overline{w}_0,\overline{w}) \in \Gamma(\bar{k})$ with $\int_0^1 \overline{w}_0(s)\, ds \leq S$, there exists $(v,w_0,w) \in \Gamma(k)$ with*

$$\int_0^1 w_0(s)\, ds \leq \int_0^1 \overline{w}_0(s)\, ds + |\bar{t} - t|$$

and $\int_0^1 |w(s)|\,ds \le \int_0^1 |\overline{w}(s)|\,ds$ *such that*

$$\|x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w}) - x_{t,x}(v,w_0,w)\|_\infty \le \bar{\bar{L}}_Q|\bar{y}-y|. \tag{2.9}$$

Proof: See Appendix 1. □

3 The Extended Problem

In this chapter we define the minimum time function T_e for the extended system (2.5). As already mentioned in the Introduction, T [resp.T_e] besides depending on (t,x), is a function of k through the constraint $\int_{\bar{t}}^{T} |\xi|\,dt \le K-k$ [resp. $\int_0^1 |w|ds \le K-k$].

Define

$$\mathcal{R}_e[\bar{k}] \doteq \Big\{(\bar{t},\bar{x}) \in \boldsymbol{R}^{1+n} : \exists (v,w_0,w) \in \Gamma(\bar{k}) \quad \text{such that} \quad x_{\bar{t},\bar{x}}(v,w_0,w)(s) \in \mathcal{T}, \text{ for some } s \in [0,1]\Big\}$$

and

$$\mathcal{R}[\bar{k}] \doteq \Big\{(\bar{t},\bar{x}) \in \boldsymbol{R}^{1+n} : \exists (v,w_0,w) \in \Gamma^+(\bar{k}) \quad \text{such that} \quad x_{\bar{t},\bar{x}}(v,w_0,w)(s) \in \mathcal{T}, \text{ for some } s \in [0,1]\Big\}.$$

$\mathcal{R}_e[\bar{k}]$ $\big[$resp.$\mathcal{R}[\bar{k}]\big]$ is the *controllable set with space-time controls* [resp. *regular space-time controls*] *having energy less than or equal to* $K-\bar{k}$. (In some literature these kinds of sets are called *reachable*).

For every $(v,w_0,w) \in \Gamma$ and every $(t,x) \in \boldsymbol{R}^{1+n}$ let us introduce

$$\theta_{t,x}(v,w_0,w) = \begin{cases} \min\{\int_0^{\bar{s}} w_0(s)\,ds \quad \bar{s} \in [0,1],\ x_{t,x}(v,w_0,w)(\bar{s}) \in \mathcal{T}\} \\ \qquad\qquad \text{if} \,\exists\, \tilde{s} \in [0,1] \,\text{such that}\, x_{t,x}(v,w_0,w)(\tilde{s}) \in \mathcal{T} \\ +\infty \qquad \text{if} \,\nexists\, \tilde{s} \in [0,1] \,\text{such that}\, x_{t,x}(v,w_0,w)(\tilde{s}) \in \mathcal{T}. \end{cases}$$

A space-time control (v,w_0,w) will be called *admissible* for (t,x) if $\theta_{t,x}(v,w_0,w) < +\infty$, and the corresponding trajectory will be called an *admissible trajectory.*

Remark 3.1. It is clear that the definition of $\theta_{t,x}$ involves, for each space-time control (v,w_0,w), only the values of s up to the first instant when the corresponding trajectory reaches the target. Hence, in view of the parameter-free character of the extended system (see Proposition 2.2), we can restrict the class of admissible controls to those whose corresponding trajectory reaches the target for $s = 1$ (obviously for these controls one has $\theta_{t,x}(v, w_0, w) = \int_0^1 w_0(s)\,ds$).

For every $(t, x, k) \in \boldsymbol{R}^{1+n} \times [0, K]$ let us define

$$T_e(t, x, k) = \inf_{(v,w_0,w)\in\Gamma(k)} \{\theta_{t,x}(v, w_0, w)\}.$$

Hence the function T_e is finite on the set

$$\mathcal{R}_{\rceil}[0, K] \doteq \bigcup_{k\in[0,K]} \mathcal{R}_{\rceil}[k] \times \{k\}$$

and it is equal to $+\infty$ on $(\boldsymbol{R}^{1+n} \times [0, K]) \setminus \mathcal{R}_e[0, K]$. In view of Remark 2.1 the function $T(t, x, k)$ can be now identified with

$$T(t, x, k) = \inf_{(v,w_0,w)\in\Gamma^+(k)} \{\theta_{t,x}(v, w_0, w)\}.$$

Setting

$$\mathcal{R}[0, K] \doteq \bigcup_{k\in[0,K]} \mathcal{R}[k] \times \{k\},$$

the function $T(t, x, k)$ is finite in $\mathcal{R}[0, K]$, while it is equal to $+\infty$ in $(\boldsymbol{R}^{1+n} \times [0, K]) \setminus \mathcal{R}[0, K]$.

In general we have $\mathcal{R}[0, K] \subseteq \mathcal{R}_e[0, K]$. The following simple example shows that unless controllability conditions are assumed (see Section 4), the inclusion is strict. Moreover it shows that at a point $(\bar{t}, \bar{x}) \in \mathcal{R}[\bar{k}]$, the optimal control (when it exists) might belong only to the extended set Γ. This, together with the density result stated in Proposition 2.4, justifies the space-time extension of the problem.

Example 3.1. Let us consider the autonomous system in $[0, 1]$

$$\begin{cases} \dot{x}_1 = r + \xi_1 \\ \dot{x}_2 = \xi_2 \\ x(0) = (x_1, x_2), \end{cases}$$

where r is a fixed positive real number. Let $\mathcal{T} = \{(0, 0)\}$, $K = 1$ and $C = \{(\xi_1, \xi_2) : 0 \le \xi_2 \le \xi_1\}$.

It is easy to see that all the admissible trajectories are contained in the cone $\{(x_1, x_2) \in \boldsymbol{R}^2 : x_1 < x_2 \le 0\}$ and that no admissible trajectory steers points of the half-line $x_1 = x_2 < 0$ to the origin. Therefore we have that $(t, (x_1, x_2)) \notin \mathcal{R}[0]$ when $x_1 = x_2$ except in the case $x_1 = x_2 = 0$. On the other hand if we consider the extended system

$$\begin{cases} t'(s) = w_0(s) \\ x_1'(s) = rw_0(s) + w_1(s) \\ x_2'(s) = w_2(s) \\ (t, x)(0) = (0, (x_1, x_2)) \end{cases} \qquad s \in [0, 1],$$

by taking the constant space-time control $(0,(1/\sqrt{2},1/\sqrt{2}))$ one shows that $\{(x_1,x_2), x_1=x_2 \text{ and } -1/\sqrt{2}\le x_1\le 0\}\subseteq \mathcal{R}_e[0]$.

Consider now $(x_1,x_2)=(-1,0)$ and the sequence of controls belonging to $W(0,0)$ given by

$$\xi_n(t)=(n-r,0)\quad \text{for } 0\le t\le 1/n.$$

The controls ξ_n are admissible, in that $x_{(-1,0)}(\xi_n)(1/n)=(0,0)$. Actually they are a minimizing sequence and $T(\bar t,(-1,0),0)=0$ for every $\bar t\in\boldsymbol{R}$. However no optimal regular control exists, while the space-time control $(0,(1,0))$ is optimal.

It is also worthwhile noting that at the points (t,x,k) where both T and T_e are finite, T might be strictly greater than T_e, as shown by the following example.

Example 3.2. Let us consider the scalar control differential equation

$$\dot x=\xi x-1$$

with $\xi\in[0,+\infty)$, $K=2$ and $\mathcal{T}=\{1,3\}$. After a trivial computation one obtains that $T_e(t,2,1)=0$ for every $t\in\boldsymbol{R}$, while $T(t,2,1)=1$.

The next two theorems establish some relations between $\mathcal{R}[k]$ and $\mathcal{R}_e[k]$, when no controllability (see Hypothesis (H) below) is assumed.

Theorem 3.1. *If $\mathcal{T}\subseteq\boldsymbol{R}^n$ is a closed set, then for every $k\in[0,K]$ we have*

$$\mathcal{R}_e[k]\subseteq\overline{\mathcal{R}}[k].$$

Proof: If $(\bar t,\bar x)\in\mathcal{R}_e[k]\setminus(\boldsymbol{R}\times\mathcal{T})$ then there exists a control $(v,w_0,w)\in\Gamma(k)$ such that

$$\left(\bar t+\int_0^1 w_0\,ds\,,\,x_{\bar t,\bar x}(v,w_0,w)(1)\right)=(\tilde t,\tilde x)\in\boldsymbol{R}\times\mathcal{T}.$$

Consider now the control $(v^-,w_0^-,w^-)(s)\doteq(v,-w_0,-w)(1-s)$. It is clear that

$$\left(\tilde t+\int_0^1 w_0^-\,ds, x_{\tilde t,\tilde x}(v^-,w_0^-,w^-)(1)\right)=(\bar t,\bar x).$$

Define

$$w_{0_n}^-=-\frac{1}{n}+w_0^-.$$

Consider the control $(v^-,w_{0_n}^-,w^-)$ and set

$$(t_n,x_n)=(\tilde t+\int_0^1 w_{0_n}^-(s)\,ds, x_{\tilde t,\tilde x}(v^-,w_{0_n}^-,w^-)(1)).$$

Clearly $\lim_{n\to+\infty}(t_n, x_n) = (\bar{t}, \bar{x})$ and indicating by $w_{0_n} \doteq -w^-_{0_n}(1-s)$ one has

$$(t_n + \int_0^1 w_{0_n}, x_{t_n,x_n}(v, w_{0_n}, w)(1)) = (\tilde{t}, \tilde{x}) \qquad \forall n \in \boldsymbol{N}.$$

This yields the thesis, in that $(v, w_{0_n}, w) \in \Gamma^+(k)$. □

Theorem 3.2. *Let $\mathcal{T} \in \boldsymbol{R}^n$ be a closed set such that either*

$$i)\mathcal{T} = \overline{\mathcal{I}\setminus\sqcup\mathcal{T}}$$
$$or$$
$$ii)\mathcal{T} = \partial\mathcal{Q} \dashv\setminus\lceil\, \mathcal{Q} = \overline{\mathcal{I}\setminus\sqcup\mathcal{Q}}$$

is verified. Then we have

$$\overline{\mathcal{R}_e[k]} = \overline{Int\mathcal{R}[k]}.$$

Proof: In view of Theorem 3.1 it is sufficient to show that $\mathcal{R}[k] \subseteq \overline{Int\mathcal{R}[k]}$. We shall prove the assertion only under hypothesis i), for the proof assuming ii) is quite similar. Let $(\bar{t}, \bar{x}) \in R[k]$ and let $(v, w_0, w) \in \Gamma(k)$ be such that $\left(\bar{t} + \int_0^1 w_0 ds, x_{\bar{t},\bar{x}}(v, w_0, w)(1)\right) = (t_1, x_1) \doteq y_1 \in \boldsymbol{R}\times\mathcal{T}$. There exist an open ball $B = B_{1+n}((\bar{t}, \bar{x}), r)$ and W neighborhood of y_1 such that the map $\phi\; B \to W$ defined by $\phi(q, z) = (q + \int_0^1 w_0 ds, x_{q,z}(v, w_0, w)(1))$ is a homeomorphism. Hence for every $\varrho < r$ there exists $(q_1, z_1) \in B((\bar{t}, \bar{x}), \varrho)$ such that $\phi(q_1, z_1) \in \boldsymbol{R}\times Int\mathcal{T}$.

Let W_1 be a neighborhood of $\phi(q_1, z_1)$ such that $W \cap W_1 \subseteq (\boldsymbol{R} \times Int\mathcal{T})$. The thesis follows from the fact that the subset $\phi^{-1}(W_1)$ is a neighborhood of (q_1, z_1), and $\phi^{-1}(W_1) \subseteq \mathcal{R}[k]$. Indeed, for every $(q, z) \in \phi^{-1}(W_1)$ there exists $\bar{s} < 1$, such that $x_{q,z}(v, w_0, w)(s) \notin Int\mathcal{T}$ for $s < \bar{s}$ and $x_{q,z}(v, w_0, w)(\bar{s}) \in \partial\mathcal{T}$. By definition of $\overline{Int\mathcal{R}[k]}$, this implies that $(\bar{t}, \bar{x}) \in \overline{Int\mathcal{R}[k]}$. □

4 Properness of the Extension and Continuity of the Minimum Time Function

In this section we prove that under a controllability condition on the dynamics of (2.1) the function T coincides with T_e and is continuous.

Let us state two Dynamic Programming Principles, the former for T and the latter for T_e. The proofs rely on standard arguments combined with obvious reparameterization techniques (see [37]). For this reason we omit them.

Dynamic Programming Principle. For every $\bar{y} = (\bar{t}, \bar{x}, \bar{k}) \in \mathcal{R}[0, K]$ one has

$$T(\bar{y}) = \inf_{(v,w_0,w)\in\Gamma^+(\bar{k})} \left\{ \int_0^s w_0(s')\,ds' + T(y(s)) \right\}, \quad \text{for every} \quad s \in [0,1], \tag{DPP}$$

where

$$y(s) = (t(s), x(s), k(s)) = \\ (\bar{t} + \int_0^s w_0(s')\,ds', x_{\bar{t},\bar{x}}(v, w_0, w)(s), \bar{k} + \int_0^s |w(s')|\,ds'),$$

while for every $\bar{y} = (\bar{t}, \bar{x}, \bar{k}) \in \mathcal{R}_e[0, K]$ one has

$$T_e(\bar{y}) = \inf_{(v,w_0,w)\in\Gamma(\bar{k})} \left\{ \int_0^s w_0(s')\,ds' + T_e(y(s)) \right\}, \quad \text{for every} \quad s \in [0,1], \tag{DPP_e}$$

where $y(s) = (t(s), x(s), k(s)) = (\bar{t} + \int_0^s w_0(s')\,ds', x_{\bar{t},\bar{x}}(v, w_0, w)(s), \bar{k} + \int_0^s |w(s')|\,ds')$.

Remark 4.1. It is clear from the character of the extension $\overline{f}$ that $\mathcal{R}[K] = \mathcal{R}_e[K]$ and that in such set we have $T_e(t, x, K) = T(t, x, K)$.

Hypothesis (H) below is the main assumption on the field at the boundary of $\mathcal{T}$. It roughly states that at each point in a neighborhood of $\partial\mathcal{T}$ and for any t there exists an ordinary control v such that the dynamics points towards $\mathcal{T}$, when the unbounded control ξ is zero. Hence it is a standard hypothesis of local controllability. If one thinks of the L^1 norm of ξ as the energy spent by the system, Hypothesis (H) can be seen as a *controllability condition with zero energy.* For standard problems, where only bounded controls appear in the dynamics, it is a refinement of former hypotheses which concerned a single point target (see [42]). In the form presented here, it is an adaptation to the non-autonomous case of a condition introduced by Cannarsa & Sinestrari in [18] (see also [25, 26, 27] and [53, 54] where similar conditions are considered). **Hypothesis (H).** For every $R > 0$ there exist $\nu_R > 0$ and $\sigma_R > 0$ such that for every $(t, x) \in (\boldsymbol{R} \times (\mathcal{T}_{\sigma_\mathcal{R}} \setminus \mathcal{T})) \cap \mathcal{B}_{\infty+\backslash}(\prime, \mathcal{R})$ there exists $v_{t,x} \in V$ such that

$$f(t, x, v_{t,x}, 0) \cdot \frac{x - \pi(x)}{|x - \pi(x)|} \leq -\nu_R$$

for some $\pi(x) \in \mathcal{T}$ such that $|x - \pi(x)| = d_\mathcal{T}(x)$.

Since $T(t,x,K)$ is nothing but the minimum time for the conventional control system $\dot{x} = f(t,x,v,0)$, Theorem 4.1 below is a slight extension of a well known result to the case where f depends on t as well and is locally Lipschitz continuous in x. Up to minor changes the proof is practically the same as in the autonomous case with global Lipschitz continuity (see [18]). We sketch it in the Appendix just for the reader's convenience.

Theorem 4.1. *Assume hypothesis* (H). *Then for all $R > 0$ there exist two positive constants C_R and δ_R, such that*

$$T(t,x,K) \leq C_R d_{\mathcal{T}}(x) \qquad \forall (t,x) \in (\boldsymbol{R} \times \mathcal{T}_{\delta_{\mathcal{R}}}) \cap \mathcal{B}_{\infty+\backslash}(\prime, \mathcal{R}). \tag{4.6}$$

In particular the set $\mathcal{R}[K]$ contains a neighborhood of $\boldsymbol{R} \times \mathcal{T}$.

Proof: See Appendix 2. □

Theorem 4.2. *Assume hypothesis* (H). *Then the sets $\mathcal{R}_e[k]$ and $\mathcal{R}[k]$ coincide and are open, for every $k \in [0,K]$.*

Proof: Let us prove that $\mathcal{R}_e[\bar{k}] \subseteq \mathcal{R}[\bar{k}]$. Let $\epsilon > 0$. If $(\bar{t},\bar{x}) \in \mathcal{R}_e[\bar{k}]$, let $(\overline{v},\overline{w}_0,\overline{w}) \in \Gamma(\bar{k})$ be such that $(\tilde{t},\tilde{x}) = (t,x)_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w})(1) \in \boldsymbol{R} \times \mathcal{T}$. Setting $\overline{w}_{0_n} = \overline{w}_0 + \frac{1}{n}$ one has $(\overline{v},\overline{w}_{0_n},\overline{w}) \in \Gamma^+(\bar{k})$ and $\|x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w}) - x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_{0_n},\overline{w})\|_\infty \leq \epsilon(n)$, with $\lim_{n\to+\infty} \epsilon(n) = 0$.

Put $t_n(s) = \bar{t} + \int_0^s \overline{w}_{0_n}(s')\,ds'$, $x_n(s) = x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_{0_n},\overline{w})(s)$ and $k(s) = \bar{k} + \int_0^s |\overline{w}|\,ds'$. If n is sufficiently large, by Theorem 4.1 there exists a control $(v,w_0,0) \in \Gamma^+(K)$ such that $\theta_{t_n(1),x_n(1)}(v,w_0,0) < +\infty$. For such a value of n, consider the control

$$(\tilde{v},\tilde{w}_0,\tilde{w})(r) = \begin{cases} (\overline{v}(2r), 2\overline{w}_{0_n}(2r), 2\overline{w}(2r)) & \text{for} \quad 0 \leq r \leq 1/2 \\ (v(2r-1), 2w_0(2r-1), 0) & \text{for} \quad 1/2 < r \leq 1. \end{cases}$$

Clearly

$$(\tilde{v},\tilde{w}_0,\tilde{w}) \in \Gamma^+(\bar{k})$$

and

$$\theta_{\bar{t},\bar{x}}(\tilde{v},\tilde{w}_0,\tilde{w}) < t_n(1) + \theta_{t_n(1),x_n(1)}(v,w_0,0) < +\infty.$$

This implies that $(\bar{t},\bar{x}) \in \mathcal{R}[\bar{k}]$.

Let us prove that $\mathcal{R}_e[\bar{k}]$ is open. Let $(\bar{t},\bar{x}) \in \mathcal{R}_e[\bar{k}]$ and $(\overline{v},\overline{w}_0,\overline{w})$ be as before. Given $\delta > 0$, by Proposition 2.5 for every $(t,x,\bar{k}) \in B((\bar{t},\bar{x},\bar{k}),\delta)$ there exists a control (v,w_0,w) in

$$N(\overline{v},\overline{w}_0,\overline{w}) = \Bigg\{ (v,w_0,w) \in \Gamma(\bar{k}) \quad \text{with}$$
$$\int_0^1 w_0\,ds \leq \int_0^1 \overline{w}_0\,ds + \delta, \quad \int_0^1 |w|\,ds \leq \int_0^1 |\overline{w}|\,ds \Bigg\}$$

and a modulus ρ such that

$$\|x_{\bar t,\bar x}(\overline{v},\overline{w}_0,\overline{w}) - x_{t,x}(v,w_0,w)\|_\infty \le \rho(\delta).$$

This implies that $(t,x)_{t,x}(v,w_0,w)(1)$ belongs to a $\rho(\delta)-$neighborhood of $\boldsymbol{R}\times\mathcal{T}$. By Theorem 4.1, for a sufficiently small δ there exists a control $(v',w_0',0) \in \Gamma^+(K)$ such that the control

$$(\tilde v,\tilde w_0,\tilde w)(r) = \begin{cases} (v(2r),2w_0(2r),2w(2r)) & \text{for} \quad 0 \le r \le 1/2 \\ (v'(2r-1),2w_0'(2r-1),0) & \text{for} \quad 1/2 < r \le 1, \end{cases}$$

(belongs to $\Gamma(\bar k)$ and) steers x to a point of $\mathcal{T}$. Hence $\theta_{t,x}(\tilde v,\tilde w_0,\tilde w) < +\infty$. It follows that $B((\bar t,\bar x,\bar k),\delta) \subseteq \mathcal{R}_e[\bar k]$, and therefore $\mathcal{R}_e[\bar k]$ is open. □

Corollary 4.1. *Assume hypothesis* (H). *We have*

$$\mathcal{R}[0,K] = \mathcal{R}_e[0,K]$$

and

$$T(t,x,k) = T_e(t,x,k)$$

for every $(t,x,k) \in \mathcal{R}[0,K]$.

Proof: The first assertion is a trivial consequence of Theorem 4.2. Let us prove that $T = T_e$ in $\mathcal{R}_e[0,K]$. Let $\epsilon > 0$. If $(\bar t,\bar x) \in \mathcal{R}_e[\bar k]$, let $(\overline{v},\overline{w}_0,\overline{w}) \in \Gamma(\bar k)$ be such that $(t,x)_{\bar t,\bar x}(\overline{v},\overline{w}_0,\overline{w})(1) \in \boldsymbol{R}\times\mathcal{T}$ and $T_e(\bar t,\bar x,\bar k) \ge \int_0^1 \overline{w}_0\, ds - \epsilon$. Set $\overline{w}_{0_n} = \overline{w}_0 + \frac{1}{n}$, $t_n(s) = \bar t + \int_0^s \overline{w}_{0_n}(s')\, ds'$, $\quad x_n(s) = x_{\bar t,\bar x}(\overline{v},\overline{w}_{0_n},\overline{w})(s)$ and $k(s) = \bar k + \int_0^s |\overline{w}|\, ds'$. In particular one has

$$\|x_{\bar t,\bar x}(\overline{v},\overline{w}_0,\overline{w}) - x_{\bar t,\bar x}(\overline{v},\overline{w}_{0_n},\overline{w})\|_\infty \le \epsilon(n)$$

with $\lim_{n\to+\infty} \epsilon(n) = 0$ and $(\overline{v},\overline{w}_{0_n},\overline{w}) \in \Gamma(\bar k)$. Since the set

$$\mathcal{K} = \bigcup_{n\in\boldsymbol{N}N} \{x_{\bar t,\bar x}(\overline{v},\overline{w}_{0_n},\overline{w})(s), \quad s \in [0,1]\} \bigcup \{x_{\bar t,\bar x}(\overline{v},\overline{w}_0,\overline{w})(s), \quad s \in [0,1]\}$$

is compact, there is an $R > 0$ such that $\mathcal{K} \subseteq B(0,R)$. By Theorem 4.1, there exist C_R and δ_R such that, if $\epsilon(\bar n) \le \delta_R$, for $n \ge \bar n$, we have

$$T_e(t_n(1),x_n(1),K) \le C_R\epsilon(n). \tag{4.7}$$

Observe that $T(t_n(1),x_n(1),k(1)) \le T(t_n(1),x_n(1),K)$. By (DPP), (4.7) and Remark 4.1, we have

$$\begin{aligned} T(\bar{t},\bar{x},\bar{k}) \leq & \int_0^1 \overline{w}_{0_n}(s)\,ds + T(t_n(1),x_n(1),k(1)) \\ \leq & \int_0^1 \overline{w}_{0_n}(s)\,ds + T(t_n(1),x_n(1),K) \\ \leq & \int_0^1 \overline{w}_0(s)\,ds + \frac{1}{n} + T_e(t_n(1),x_n(1),K) \\ \leq & T_e(\bar{t},\bar{x},\bar{k}) + \frac{1}{n} + C_R\epsilon(n) + \epsilon. \end{aligned} \tag{4.8}$$

Taking the limit as $n \to +\infty$ on the r.h.s. of (4.8), we get, by the arbitrariness of ϵ, that $T(\bar{t},\bar{x},\bar{k}) \leq T_e(\bar{t},\bar{x},\bar{k})$. The opposite inequality is obvious. □

Observe that the proof of Corollary 4.1 implies that every admissible trajectory of the extended system (2.5) can be approximated with *regular* admissible trajectories. This fact and Corollary 4.1 say that the introduction of space-time controls is an actual *extension* of the original problem (see [56]).

In what follows we always assume the controllability Hypothesis (H). Hence, in view of Corollary 4.1, we can identify T with T_e and $\mathcal{R}[0,K]$ with $\mathcal{R}_e[0,K]$.

In order to prove the continuity of T we begin by showing that it is locally bounded.

Lemma 4.1. *Assume hypothesis* (H)*. Then the function T is bounded on compact subsets of* $\mathcal{R}[0,K]$.

Proof: Let $\bar{y} = (\bar{t},\bar{x},\bar{k}) \in \mathcal{R}[\bar{k}]$. Consider $B(\bar{y},\delta)$ for some positive δ. Let $(\overline{v},\overline{w}_0,\overline{w}) \in \Gamma(\bar{k})$ be such that $x(1) = x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w})(1) \in \mathcal{T}$ and $T(\bar{t},\bar{x},\bar{k}) \geq \int_0^1 \overline{w}_0(s) - \epsilon$. In view of Proposition 2.5, for every $y = (t,x,k) \in B(\bar{y},\delta)$, there exists a control in the set $N(\overline{v},\overline{w}_0,\overline{w})$ defined in the proof of Theorem 4.2 such that $\|x_{\bar{t},\bar{x}}(\overline{v},\overline{w}_0,\overline{w}) - x_{t,x}(v,w_0,w)\|_\infty \leq \rho(\delta)$, where ρ is a suitable modulus.

Choose R such that $B_n(0,R) \supset \{x_{t,x}(v,w_0,w)(s),\ s \in [0,1], \forall (t,x,k) \in B(\bar{y},\delta),\ \forall (v,w_0,w) \in N(\overline{v},\overline{w}_0,\overline{w})\}$. Let δ_R and C_R be as in Theorem 4.1. It is not restrictive to assume that $\rho(\delta) \leq \delta_R$. Then we have

$$\begin{aligned} T(y) \leq & \int_0^1 w_0(s)\,ds + C_R\rho(\delta) \\ \leq & \int_0^1 \overline{w}_0(s)ds + \delta + C_R\rho(\delta) \leq T(\bar{y}) + \delta + C_R\delta_R + \epsilon, \end{aligned}$$

for every $y \in B(\bar{y}, \delta) \cap \mathcal{R}[0, K]$. A trivial compactness argument concludes the proof. □

Theorem 4.3. *Assume hypothesis* (H)*. Then the function* $T : \boldsymbol{R}^{1+n} \times [0, K] \to \boldsymbol{R} \cup \{+\infty\}$ *is continuous in* $\overline{\mathcal{R}[0, K]}$.

Let Q be a compact subset of $\mathcal{R}[0, K]$. We shall prove that T is uniformly continuous on Q. By the previous lemma there exists S such that $T(Q) \leq S$. Hence, on Q, T coincides with the value map $\tilde{T}$ of the problem corresponding to controls (v, w_0, w) such that $\int_0^1 w_0(s)\, ds \leq S + 1$, i.e.,

$$T(t, x, k) = \tilde{T}(t, x, k) \doteq \inf_{\substack{(v, w_0, w) \in \Gamma(k) \\ \int_0^1 w_0 \leq S+1}} \{\theta_{t,x}(v, w_0, w)\}.$$

In particular we can apply Proposition 2.5 to this subclass of controls. Let $\epsilon > 0$ and let us consider $(t_i, x_i, k_i) \in Q$, $i = 1, 2$, such that $|(t_1, x_1, k_1) - (t_2, x_2, k_2)| \leq \frac{\epsilon}{2}$. Let us choose a space-time control $(v_1, w_{0_1}, w_1) \in \Gamma(k_1)$ (with $\int_0^1 w_{0_1} \leq S + 1$) verifying $x_{t_1,x_1}(v_1, w_{0_1}, w_1)(s) \notin \mathcal{T} \quad \forall \int \in [\prime, \infty)$, $x_{t_1,x_1}(v_1, w_{0_1}, w_1)(1) \in \mathcal{T}$, and $\int_0^1 w_{0_1} \leq T(t_1, x_1, k_1) + \frac{\epsilon}{2}$. By Proposition 2.5 there is a control $(v_2, w_{0_2}, w_2) \in \Gamma(k_2)$ (with $\int_0^1 w_{0_2} \leq S+1$) such that $\int_0^1 w_{0_2} \leq \int_0^1 w_{0_1} + \frac{\epsilon}{2}$ and $|x_{t_1,x_1}(v_1, w_{0_1}, w_1)(1) - x_{t_2,x_2}(v_2, w_{0_2}, w_2)(1)| \leq \rho(\frac{\epsilon}{2})$ where ρ is a suitable modulus.

Let $B_{1+n}[0, R]$ be so large to contain all the trajectories issuing from Q and corresponding to controls (v, w_0, w) such that $\int_0^1 w_0 \leq S+1$. Let δ_R and C_R be the corresponding constants whose existence is stated in Theorem 4.1. By taking ϵ sufficiently small we obtain $x_{t_2,x_2}(v_2, w_{0_2}, w_2)(1) \in \mathcal{T}_{\delta_{\mathcal{R}}}$. Hence in view of (DPP) and of Theorem 4.1 we obtain

$$\begin{aligned}
\tilde{T}(t_2, x_2, k_2) &\leq \int_0^1 w_{0_2} + \tilde{T}\left((t, x)_{t_2,x_2}(v_2, w_{0_2}, w_2)(1), k_2 + \int_0^1 |w_2(s)|\, ds\right) \\
&\leq \int_0^1 w_{0_1} + \frac{\epsilon}{2} + \tilde{T}\left((t, x)_{t_2,x_2}(v_2, w_{0_2}, w_2)(1), K\right) \\
&\leq \tilde{T}(t_1, x_1, k_1) + \epsilon + C_R \rho(\frac{\epsilon}{2}).
\end{aligned}$$

By interchanging the roles of (t_1, x_1, k_1) and (t_2, x_2, k_2) we obtain

$$|\tilde{T}(t_1, x_1, k_1) - \tilde{T}(t_2, x_2, k_2)| \leq \epsilon + C_R \rho(\frac{\epsilon}{2})$$

as soon as $|(t_1, x_1, k_1) - (t_2, x_2, k_2)| \leq \frac{\epsilon}{2}$.

To conclude the proof we must show that if $(y_n)_{n \in \boldsymbol{N}} = (t_n, x_n, k_n)$ is a sequence belonging to $\mathcal{R}[0, K]$ and $\lim_{n \to +\infty} y_n = \bar{y} = (\bar{t}, \bar{x}, \bar{k}) \in \overline{\mathcal{R}[0, K]} \setminus \mathcal{R}[0, K]$ then we have $\lim_{n \to +\infty} T(y_n) = +\infty$. If not, there would be a subsequence, still denoted by (y_n), such that $\lim_{n\to+\infty} T(y_n) = T$. Choose a sequence of minimizing controls (v_n, w_{0_n}, w_n) such that $\int_0^1 w_{0_n}(s)\,ds \leq T(y_n) + \frac{1}{n}$ and $x_{t_n,x_n}(v_n, w_{0_n}, w_n)(1) \in \mathcal{T}$. It is trivial to check that the corresponding trajectories verify that

$$\lim_{n\to+\infty} \|x_{\bar{t},\bar{x}}(v_n, w_{0_n}, w_n) - x_{t_n,x_n}(v_n, w_{0_n}, w_n)\|_\infty = 0.$$

With an argument analogous to the one exploited in the first part of the proof we obtain that $(\bar{t}, \bar{x}, \bar{k}) \in \mathcal{R}[0, K]$, a contradiction. □

The result can be improved as soon as one assumes Hypothesis (L) introduced in section 2:

Corollary 4.2. *Assume hypotheses* (L) *and* (H). *Then the function T is locally Lipschitz continuous in $\mathcal{R}[0, K]$.*

Proof: Let $Q \subseteq \mathcal{R}[0, K]$ be a compact subset. Then there exists $S > 0$ such that $T(t, x, k) \leq S$, for every $(t, x, k) \in Q$. The set of trajectories from points of Q is compact if we use space-time controls (v, w_0, w) such that $\int_0^1 w_0(s)\,ds \leq S$. Proceeding as in the proof of the continuity of T it is easy to prove that (4.6) together with (2.9) (which now replaces (2.8)) implies the local Lipschitz continuity of T. □

5 Hamilton-Jacobi Equation

The aim of this section is to recover the value map T as the unique solution of a suitable boundary value problem. Because of the sublinearity in the unbounded control the Hamiltonian turns out to be equal to $-\infty$ at several points. Hence, analogously to what has been done for the Boltz problem in [37], [39, 40], we regularize the problem by considering a continuous Hamiltonian which is naturally connected with the extended control system.

Let us consider the Hamiltonian

$$H(t, x, p_t, p_x, p_k) \doteq \min_{v\in V, (w_0,w)\in S_m^+} \mathcal{H}(t, x, p_t, p_x, p_k, v, w_0, w),$$

where

$$\mathcal{H}(t, x, p_t, p_x, p_k, v, w_0, w) \doteq w_0 + p_t w_0 + p_x \cdot \overline{f}(t, x, v, w_0, w) + p_k |w|,$$

with

$$S_m^+ = S_m \cap \{[0, +\infty) \times C\} \quad \text{and} \quad S_m = \{(w_0, w) \in \boldsymbol{R}^{m+1} : \|w_0, w\| = 1\}.$$

Notice that, unlike the formal Hamiltonian of the problem, H is continuous. The reason of that relies on the fact that we have replaced a minimization over an unbounded set (the cone $[0,+\infty)\times C$) with a minimization over the compact set S_m^+. As it will be clear later, the fact that S_m^+ does not contain the origin is essential in order to prove the uniqueness of the corresponding boundary value problem (see also [37]).

Let us set $\nabla T \doteq (\nabla_t T, \nabla_x T, \nabla_k T)$, where $\nabla_t T, \nabla_x T, \nabla_k T$ denote the gradients of T with respect to t, x and k, respectively. We will prove that T is a viscosity solution of the Hamilton- Jacobi equation

$$-H(t,x,\nabla T(t,x,k)) = 0 \tag{HJ}$$

in the open set

$$\Omega \doteq \mathcal{R}(0,K)\cap(\boldsymbol{R}\times(\boldsymbol{R}^n\setminus\mathcal{T})\times(\imath,\mathcal{K}))\,,$$

where $\mathcal{R}(0,K)\doteq\bigcup_{k\in(0,K)}\mathcal{R}[k]\times\{k\}$. Moreover we shall establish boundary conditions on the sets:

$$\partial\Omega_0 \doteq (\mathcal{R}[0]\setminus(\boldsymbol{R}\times\mathcal{T}))\times\{\imath\}$$

and

$$\partial\Omega_K \doteq (\mathcal{R}[K]\setminus(\boldsymbol{R}\times\mathcal{T}))\times\{\mathcal{K}\}.$$

For the reader convenience we recall the notion of viscosity solution (see e.g. [21]).

Definition 5.1. *Let E be a subset of $\boldsymbol{R}^{n+2}$.*

A function $\nu\in C^0(E)$ is a viscosity subsolution *of (HJ) at $(t,x,k)\in E$ if for any $\varphi\in C^\infty(\boldsymbol{R}^{n+2})$ such that (t,x,k) is a local maximum of $\nu-\varphi$ on E one has*

$$-H(t,x,\nabla\varphi(t,x,k))\le 0.$$

$\nu\in C^0(E)$ is a viscosity supersolution *of (HJ) at $(t,x,k)\in E$ if for any $\varphi\in C^\infty(\boldsymbol{R}^{n+2})$ such that (t,x,k) is a local minimum of $\nu-\varphi$ on E one has*

$$-H(t,x,\nabla\varphi(t,x,k))\ge 0.$$

$\nu\in C^0(E)$ is a viscosity solution *of (HJ) at $(t,x,k)\in E$ if it is both a viscosity subsolution and a viscosity supersolution.*

Theorem 5.1. *Assume hypothesis* (H). *Then*

i) T is a viscosity solution of (HJ) in $\Omega\cup\partial\Omega_0$,

ii) T is a viscosity supersolution of (HJ) on $\partial\Omega_K$.

Proof: The proof of this result, besides involving some standard arguments, is mainly based on the fact that $\overline{f}$ is homogeneous in (w_0,w) and that we can canonically parameterize the space-time controls. Let us show that T is a subsolution of (HJ) in $\Omega \cup \partial\Omega_0$. Let $\bar{y} = (\bar{t}, \bar{x}, \bar{k}) \in \Omega \cup \partial\Omega_0$ and let φ be a function of $\mathcal{C}^\infty(\boldsymbol{R}^{n+2})$, such that

$$T(y) - \varphi(y) \leq T(\bar{y}) - \varphi(\bar{y}),$$

for every $y = (t, x, k)$ in $(\Omega \cup \partial\Omega_0) \cap B(\bar{y}, r_0)$, where r_0 is sufficiently small. Let $w = (w_0, w_1, \ldots, w_m) \in S^+_m[0,1]$ and $v \in V$. Then the control

$$(\overline{v}, \overline{w}_0, \overline{w})(s) = \begin{cases} (v, w_0, w)(s), & \text{for} \quad 0 \leq s \leq \epsilon \\ (v, w_0, 0)(s), & \text{for} \quad \epsilon < s \leq 1, \end{cases}$$

belongs to $\Gamma(\bar{k})$ as soon as we choose $\epsilon > 0$ sufficiently small. Moreover setting $\bar{y}(s) = (\bar{t} + \int_0^s \overline{w}_0 \, d\sigma, x_{\bar{t},\bar{x}}(\overline{v}, \overline{w}_0, \overline{w})(s), \bar{k} + \int_0^s |\overline{w}| \, d\sigma)$, there exists $\bar{s} > 0$ such that $\bar{y}(s) \in B(\bar{y}, r_0)$ for each $s \in [0, \bar{s}]$. By (DPP) we have

$$\frac{\varphi(\bar{y}) - \varphi(\bar{y}(s))}{s} \leq \frac{T(\bar{y}) - T(\bar{y}(s))}{s} \leq \frac{\int_0^s \overline{w}_0(s)\, ds}{s}.$$

Taking the limit as $s \to 0$ we get

$$\nabla_t \varphi(\bar{y}) + \nabla_x \varphi(\bar{y}) \cdot \overline{f}(\bar{t}, \bar{x}, v, w_0, w) + \nabla_k \varphi(\bar{y}) |w| \geq -w_0$$

and since (v, w_0, w) was arbitrary in $V \times S^+_m$ we deduce that

$$-\inf_{v \in V, (w_0, w) \in S^m_+} \mathcal{H}(\bar{t}, \bar{x}, \nabla_t \varphi(\bar{y}), \nabla_x \varphi(\bar{y}), \nabla_k \varphi(\bar{y}), v, w_0, w) \leq 0.$$

We prove now that T is a supersolution in $\Omega \cup \partial\Omega_0 \cup \partial\Omega_K$. Let $(\bar{t}, \bar{x}, \bar{k}) = \bar{y}$ and let φ be a map in $C^\infty(\boldsymbol{R}^{n+2})$ such that $\bar{y}$ is a local minimum for $T - \varphi$ on $\Omega \cup \partial\Omega_0 \cup \partial\Omega_K$. We can suppose that $T(\bar{y}) = \varphi(\bar{y})$. Hence there exists r_0 such that $T(y) - \varphi(y) \geq 0$ is verified for every $y \in B(\bar{y}, r_0) \cap (\Omega \cup \partial\Omega_0 \cup \partial\Omega_K)$.

By (DPP) for every s we can choose a sequence $(v_n, w_{0_n}, w_n) \in \Gamma(\bar{k})$ such that denoting by $y_n(s) = (t_n(s), x_n(s), k_n(s)) = (\bar{t} + \int_0^s w_{0_n}(s')\, ds',$ $x_{\bar{t},\bar{x}}(v_n, w_{0_n}, w_n)(s), \bar{k} + \int_0^s |w_n(s')|\, ds')$, we have $\int_0^s w_{0_n}(s')\, ds' + T(y_n(s)) \leq T(\bar{y}) + \frac{1}{n^2}$. We can suppose $x_n(1) = x_{\bar{t},\bar{x}}(v_n, w_{0_n}, w_n)(1) \in \mathcal{T}$ and $x_n(s) \notin \mathcal{T}$ for every $s < 1$. By (2.7) there exists $\bar{s} \in [0,1]$ such that for every $s \leq \bar{s}$ and every $n \in \boldsymbol{N}$ we have $y_n(s) \in B(\bar{y}, r_0)$ and therefore

$$\begin{aligned} \varphi(y_n(s)) + \int_0^s w_{0_n}(s')\, ds' &\leq T(y_n(s)) + \int_0^s w_{0_n}(s')\, ds' \\ &\leq T(\bar{y}) + \frac{1}{n^2} = \varphi(\bar{y}) + \frac{1}{n^2}. \end{aligned} \tag{5.1}$$

For an n sufficiently large, set $s = \frac{1}{n}$ in (5.1). Then one has

$$\int_0^{\frac{1}{n}} w_{0_n}(s')\, ds' + \varphi(y_n(\frac{1}{n})) - \varphi(\bar{y}) \leq \frac{1}{n^2},$$

that is

$$\int_0^{\frac{1}{n}} \mathcal{H}(x_n(s), t_n(s), \nabla(\varphi(t_n(s), x_n(s), k_n(s)), v_n(s), w_{0_n}(s), w_n(s))\, ds \leq \frac{1}{n^2}. \tag{5.2}$$

It is not restrictive to assume that the space-time controls (v_n, w_{0_n}, w_n) are canonically parameterized (see Proposition 2.3), which implies that

$$|w_{0_n}, w_n|(s) = \int_0^1 |(w_{0_n}, w_n)(\sigma)|\, d\sigma \qquad \forall s \in [0,1].$$

We shall prove that there exists $A > 0$ such that

$$\int_0^1 |(w_{0_n}, w_n)|(\sigma)\, d\sigma \geq A \tag{5.3}$$

for all $n \in \boldsymbol{N}$. Since all the functions are continuous, taking the limit as $s \to 0$ inside the integral of (5.2), we get that there exists a sequence $\epsilon(n)$ with $\lim_{n\to+\infty} \epsilon(n) = 0$ such that

$$\begin{aligned}\epsilon(n) &\geq n \int_0^{\frac{1}{n}} \mathcal{H}(\bar{t}, \bar{x}, \nabla\varphi(\bar{t}, \bar{x}, \bar{k}), v_n(\sigma), w_{0_n}(\sigma), w_n(\sigma))\, d\sigma \\ &\geq n \int_0^1 |(w_{0_n}, w_n)|(\sigma)\, d\sigma \int_0^{\frac{1}{n}} \min_{\substack{(w_0,w)\in S_m^+ \\ v\in V}} \mathcal{H}(\bar{t}, \bar{x}, \nabla\varphi(\bar{t}, \bar{x}, \bar{k}), v, w_0, w)\, d\sigma \\ &\geq A\, H(\bar{t}, \bar{x}, \nabla\varphi(\bar{t}, \bar{x}, \bar{k})).\end{aligned}$$

The above expression gives the required inequality once we consider the limit of both sides as $n \to +\infty$.

Let us conclude by proving the claim that (5.3) holds true. If for every s and every sequence of the above controls (v_n, w_{0_n}, w_n) (5.3) does not hold, then there would be a subsequence, still denoted by (v_n, w_{0_n}, w_n) such that $\lim_{n\to+\infty} \int_0^1 |(w_{0_n}, w_n)|(\sigma)\, d\sigma = 0$. Since all the trajectories $(t_n, x_n)_{\bar{t},\bar{x}}$, are contained in a compact set Q, by setting $L = L_Q$, we have

$$\|(t_n, x_n)_{\bar{t},\bar{x}} - (\bar{t}, \bar{x})\|_\infty \leq M_{\bar{t},\bar{x}} L^{-1} \left(e^{L \int_0^1 |w_{0_n}(s), w_n(s)|\, ds} - 1 \right)$$

where $M_{\bar{t},\bar{x}}$ is the constant appearing after Lemma 2.1.

In particular we would obtain $\bar{x} \in \mathcal{T}$, a contradiction. □

We are now in position to provide an uniqueness theorem. We will use a comparison theorem due to M. Bardi and P. Soravia (see [5], [6]) which gives uniqueness of viscosity solutions that satisfy mixed boundary conditions: that is, Dirichlet conditions and conditions of sub or super viscosity solution, also called *constrained* conditions at the boundary of Ω. In our case T is a viscosity supersolution of (HJ) on $\partial\Omega_K$, while it satisfies classical Dirichlet conditions on $\partial\mathcal{T}$. We will make use of a Kruskov-type transform which allows to convert the condition that on $\overline{\mathcal{R}[\prime,\mathcal{K}]} \setminus \mathcal{R}[\prime,\mathcal{K}]$ the minimum time function is $+\infty$ into a classical Dirichlet condition.

Finally we have to remark that the main difference with a system with bounded controls is that the function T is allowed to be equal to zero even at points that do not belong to the boundary of the target.

Theorem 5.2. *Assume hypothesis* (H). *Let f be locally Lipschitz continuous in (t,x) and such that $|\overline{f}(t,x,v,w_0,w)| \leq \bar{C}(1+|(t,x)|)(w_0+|w|)$. Then T is the unique, continuous and lower bounded function in $\mathcal{R}[0,K]$, which is a viscosity solution of (HJ) in Ω and satisfies the following boundary conditions:*

$$\begin{gathered} T(y) = 0 \qquad \forall y \in \boldsymbol{R} \times \partial\mathcal{T} \times [\prime,\mathcal{K}], \\ T(y) = +\infty \qquad \forall y \in \overline{\mathcal{R}[\prime,\mathcal{K}]} \setminus \mathcal{R}[\prime,\mathcal{K}], \\ T \textit{ is a viscosity supersolution of (HJ) on} \partial\Omega_K, \\ T \textit{ is a viscosity solution of (HJ) on} \partial\Omega_0. \end{gathered} \tag{BC}$$

Proof: Let us consider the map

$$S(t,x,k) = \begin{cases} 1 - e^{-T(t,x,k)+k} & \text{for} \quad (t,x,k) \in \mathcal{R}[0,K], \\ 1 & \text{for} \quad (t,x,k) \in (\boldsymbol{R}^{1+n} \times [0,K]) \setminus \mathcal{R}[0,K]. \end{cases}$$

By the second boundary condition, S turns out to be continuous in $\boldsymbol{R}^{1+n} \times [0,K]$. Note that S is bounded and it is straightforward to prove that S is a viscosity solution of

$$\begin{aligned} S - \min_{\substack{(w_0,w)\in S_m^+ \\ v\in V}} (1 + \nabla_t S \frac{w_0}{w_0+|w|} &+ \nabla_x S \cdot \frac{\overline{f}(t,x,v,w_0,w)}{w_0+|w|} \\ &+ \nabla_k S \frac{|w|}{w_0+|w|}) = 0 \end{aligned} \tag{5.5}$$

in $\boldsymbol{R}^{1+n} \times [0,K] \setminus (\boldsymbol{R} \times \mathcal{T} \times [\prime,\mathcal{K}])$.

We introduce the Hamiltonian

$$F(t,x,r,p_t,p_x,p_k) \doteq r + \max_{\substack{(w_0,w)\in S_m^+ \\ v\in V}} \mathcal{F}(t,x,p_t,p_x,p_k,v,w_0,w)$$

where

$$\mathcal{F}(t,x,p_t,p_x,p_k,v,w_0,w) = -(1+p_t\frac{w_0}{w_0+|w|}+ \\ p_x\cdot\frac{\overline{f}(t,x,v,w_0,w)}{w_0+|w|}+p_k\frac{|w|}{w_0+|w|})$$

so that we can write (5.5) as

$$F(t,x,S,\nabla S)=0. \qquad \text{(HJ')}$$

The boundary conditions (BC) are transformed into

$$\begin{aligned} &S(t,x,k)=1-e^k \qquad \text{for} \quad (t,x,k)\in \boldsymbol{R}\times\partial\mathcal{T}\times[\prime,\mathcal{K}],\\ &S \quad \text{is a viscosity supersolution of (HJ') in} \quad \boldsymbol{R}^{1+n}\times\{K\}, \qquad \text{(BC')}\\ &S \quad \text{is a viscosity solution of (HJ') in} \quad \boldsymbol{R}^{1+n}\times\{0\}. \end{aligned}$$

In order to apply the comparison theorem of [6] we have to prove that, given a compact $Q \subset \boldsymbol{R}^{1+n}$, setting

$$p=(p_t,p_x,p_k)$$

and

$$q=(q_t,q_x,q_k)\in R^{n+2},$$

there exists a constant C such that

$$|F(t_1,x_1,r,p)-F(t_2,x_2,r,q)|\le C\,(1+|(t_1,x_1)|)\,|p-q|+ \\ L_Q|q||(t_1,x_1)-(t_2,x_2)|$$

for every $(t_1,x_1,r,p),(t_1,x_2,r,q)\in Q\times\boldsymbol{R}^{n+3}$. The sublinear growth of f in (t,x) implies that $|\overline{f}(t,x,v,w_0,w)|\le\bar{C}(1+|(t,x)|)(w_0+|w|)$. Choose $(v,w_0,w)\in V\times S_m^+$ such that $F(t_2,x_2,r,q)=r+\mathcal{F}(t_2,x_2,q,v,w_0,w)$. We have

$$\begin{aligned} &F(t_2,x_2,r,q)-F(t_1,x_1,r,p)\le\frac{w_0}{w_0+|w|}|p_t-q_t|+\frac{|w|}{w_0+|w|}|p_k-q_k|+\\ &+|\,\overline{f}(t_1,x_1,v,w_0,w)\cdot\frac{p_x}{w_0+|w|}-\overline{f}(t_2,x_2,v,w_0,w)\cdot\frac{q_x}{w_0+|w|}\,|\\ &\le L_Q|(t_1,x_1)-(t_2,x_2)||q_x|+\bar{C}(1+|t_1,x_1|)|p_x-q_x|+2|p-q|\\ &\le L_Q|(t_1,x_1)-(t_2,x_2)||q|+C(1+|t_1,x_1|)|p-q| \end{aligned}$$

where $C=\bar{C}+2$.

The inequality holds true in absolute value once we exchange (t_1,x_1) with (t_2,x_2). We apply Theorem 1.2 of [6] and Corollary 1.5 to get uniqueness for the solution of (HJ') that satisfies (BC'). Using the inverse transformation $T(t,x,k)=k-log(1-S(t,x,k))$ we recover uniqueness for the solution of (HJ) and (BC).

We point out that Theorem 1.2 in [6] is proved under a condition weaker than Lipschitz continuity on (t,x) but of global nature. It is not difficult to prove that the theorem still holds if we assume local Lipschitz continuity and sublinear growth on (t,x). $\square$

Appendix 1

Here we present the proof of Proposition 2.5, which is a variation of Theorem 3.1 in [39]. We need some definitions and notations. Fix a compact $Q \subset \boldsymbol{R}^{1+n}$ and $S>0$. Let us introduce

$$\begin{aligned}\phi(w) = \max\Big\{ &|\overline{f}(y,v,w_0,w) - \overline{f}(y,v,w_0,0)|,\\ &\text{for} \quad (y,v,w_0) \in Q \times V \times [0,S+K]\Big\}\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}\psi(w_0) = \max\Big\{ &|\overline{f}(y,v,\overline{w}_0+w_0,w) - \overline{f}(y,v,\overline{w}_0,w)|,\\ &\text{for} \quad (y,v,\overline{w}_0,w) \in Q \times V \times B_{1+m}[0,S+K]\Big\}.\end{aligned} \tag{A.2}$$

Lemma A.1. *Let* $w_0(\cdot) \in L^1([0,1],[0,S+K])$ *and* $w \in L^1([0,1],B_m[0,S+K])$. *Then* $\phi \circ w(\cdot)$ *and* $\psi \circ w_0(\cdot)$ *belong to* $L^1([0,1],\boldsymbol{R})$ *and there exist two modulus* $\overline{\phi}(\cdot)$ *and* $\overline{\psi}(\cdot)$, *defined in* $[0,S+K]$, *such that*

$$\int_0^1 \phi(w(s))\,ds \le \overline{\phi}\Big(\int_0^1 |w|(s)\,ds\Big) \quad \text{and} \quad \int_0^1 \psi(w_0)\,ds \le \overline{\psi}\Big(\int_0^1 w_0(s)\,ds\Big).$$

Moreover if we assume hypothesis (L) *then we have*

$$\int_0^1 \phi(w(s))\,ds \le N_Q \int_0^1 |w|(s)\,ds \quad \text{and} \quad \int_0^1 \psi(w_0)\,ds \le N_Q \int_0^1 w_0(s)\,ds,$$

for a suitable N_Q *depending only on* Q.

Proof: Indicate by μ the modulus of continuity of the restriction of $\overline{f}$ to $Q \times V \times B_{1+m}[0,S+K]$ with respect to the w variable and by μ_0 the one with respect to the w_0 variable. Then we have $|\phi(w_1)-\phi(w_2)| \le \mu(|w_1-w_2|)$ and $|\psi(w_{0_1})-\psi(w_{0_2})| \le \mu_0(|w_{0_1}-w_{0_2}|)$. The map $w \to \mu(|w|)$ is bounded, continuous at $w=0$ and verifies $\mu(0)=0$. Hence it is well known that the superposition operator $w \to \mu \circ w$ acts from $L^1([0,1],B_m[0,S+K])$ with values in $L^1([0,1],\boldsymbol{R})$ and it is continuous at zero. Therefore $\int_0^1 |w_1-w_2| \le$

δ yields $\int_0^1 |\phi(w_1) - \phi(w_2)| \leq \int_0^1 \mu(|w_1 - w_2|) \leq \epsilon$. This implies that the operator $w \to \phi \circ w$, defined in $L^1([0,1], B_m[0, S+K])$ with values in $L^1([0,1], \boldsymbol{R})$ is uniformly continuous. If we define $\overline{\phi}$ to be its modulus of continuity, we get the first inequality of the thesis. Analogously we obtain the second inequality, as soon as $\overline{\psi}$ is the modulus of continuity of the operator $w_0 \to \psi \circ w_0$.

It is clear that if hypothesis (L) is assumed, then $\mu(|w_1 - w_2|)$ and $\mu_0(|w_{0_1} - w_{0_2}|)$ can be replaced by $N_Q|w_1 - w_2|$ and $N_Q|w_{0_1} - w_{0_2}|$, respectively, for a suitable N_Q. □

Proof of Proposition 2.5: If $t > \bar{t}$ define $\bar{s} \doteq \min\{s \in [0,1] : \bar{t} + \int_0^s \overline{w}_0 \, d\sigma = t\}$and

$$w_0(s) = \begin{cases} 0 & \text{for} \quad s \in [0, \bar{s}] \\ \overline{w}_0(s) & \text{for} \quad s \in (\bar{s}, 1] \end{cases}$$

while if $t \leq \bar{t}$ define

$$w_0(s) = \overline{w}_0(s) + (\bar{t} - t).$$

Set $\bar{t}(s) = \bar{t} + \int_0^s \overline{w}_0(\sigma) \, d\sigma$ and $t(s) = t + \int_0^s w_0(\sigma) \, d\sigma$. Notice that in both cases we have $|\bar{t}(s) - t(s)| \leq |\bar{t} - t|$.

Now define $\bar{\bar{s}} \doteq \max\left\{ s \in [0,1] : \int_0^s |\overline{w}(\sigma)| \, d\sigma \leq K - k \right\}$ and introduce

$$w(s) = \begin{cases} \overline{w}(s) & \text{for} \quad s \in [0, \bar{\bar{s}}] \\ 0 & \text{for} \quad s \in (\bar{\bar{s}}, 1]. \end{cases}$$

This implies that $\overline{w}(s) = w(s)$ in the case $k \leq \bar{k}$. Notice that $\int_0^1 |\overline{w} - w| \, d\sigma \leq |\bar{k} - k|$ and $\int_0^1 |w| \, d\sigma \leq \int_0^1 |\overline{w}| \, d\sigma$. Finally define $\overline{x}(s) \doteq x_{\bar{t},\bar{x}}(\overline{v}, \overline{w}_0, \overline{w})(s)$ and $x(s) \doteq x_{t,x}(v, w_0, w)(s)$. If $(t, x, k) \in B_{n+2}(\bar{y}, \delta)$, these trajectories are contained in a compact set $Q' \subset \boldsymbol{R}^{1+n}$. Let us set $L = L_{Q'}$, $M = \max\limits_{Q' \times V \times [0,1] \times (B_m[0,1] \cap C)} |\overline{f}|$, and let us consider the maps ϕ and ψ introduced in (A.1) and (A.2), with $S = \int_0^1 \overline{w}(s) \, ds + 1$ and Q replaced by Q'. The following estimates hold (see Lemma 3.1 of [39] for details): if $t > \bar{t}$

$$\begin{aligned} &|\overline{x}(s) - x(s)| \leq |\bar{x} - x| + \int_0^{\bar{s}} \psi(w_0(\sigma)) \, d\sigma + M(|\bar{t} - t| + |\bar{k} - k|) + \\ &+ \int_{\bar{\bar{s}}}^1 \phi(w(\sigma)) \, d\sigma + \omega(|\bar{t} - t|) \int_0^1 (\overline{w}_0(\sigma) + |\overline{w}(\sigma)|) \, d\sigma + \\ &+ L \int_0^s (\overline{w}_0(\sigma) + |\overline{w}(\sigma)|) |\overline{x}(\sigma) - x(\sigma)| \, d\sigma \end{aligned}$$

while, if $t \leq \bar{t}$,

$$\begin{aligned} &|\overline{x}(s) - x(s)| \leq |\bar{x} - x| + \int_{\bar{\bar{s}}}^1 \phi(w(\sigma)) \, d\sigma + \int_0^{\bar{s}} \psi(\bar{t} - t) \, d\sigma + \\ &+ \omega(|\bar{t} - t|) \int_0^1 (w_0(\sigma) + |\overline{w}(\sigma)|) \, d\sigma \\ &+ L \int_0^s (w_0(\sigma) + |\overline{w}(\sigma)|) |\overline{x}(\sigma) - x(\sigma)| \, d\sigma. \end{aligned}$$

In both cases using Gronwall's lemma we get

$$|\overline{x}(s) - x(s)| \leq \Big(|\bar{x} - x| + M(|\bar{t} - t| + |\bar{k} - k|) + \overline{\psi}(|\bar{t} - t|) + \overline{\phi}(|\bar{k} - k|) + \\ + \omega(|\bar{t} - t|) \int_0^1 (\overline{w}_0(\sigma) + |\overline{w}(\sigma)| + |\bar{t} - t|)\, d\sigma \Big) e^{L \int_0^s (\overline{w}_0(\sigma) + |\overline{w}(\sigma)| + |\bar{t} - t|)\, d\sigma}, \tag{A.3}$$

where $\overline{\psi}(\cdot)$ and $\overline{\phi}(\cdot)$ are the same as in Lemma A.1. Hence we obtain (2.8) for a suitable choice of ρ. Moreover in view of the above arguments the second part of the thesis is trivial. □

Proof of Corollary 2.1: If we assume hypothesis (L), the estimate (A.3) yields (2.9) since, by the last part of Lemma A.1, we can replace $\overline{\psi}(|\bar{t} - t|)$ and $\overline{\phi}(|\bar{k} - k|)$ by $\bar{N}_{Q'}|\bar{t} - t|$ and $\bar{N}_{Q'}|\bar{k} - k|$, respectively. Taking the value of the r.h.s. of (A.3) for $s = 1$ we get (2.9). □

Appendix 2

Proof of Theorem 4.1. As mentioned in Section 4, this proof mimics the proof of Proposition 2.2 in [18], where the autonomous case with global Lipschitz continuity, is investigated. We outline it just for the sake of self-consistency. We first prove (4.6) under the additional assumption that $\mathcal{T}$ is bounded, that f is globally Lipschitz continuous with respect to x with constant L, and that there exists $M > 0$ such that $|f(t, x, v, 0)| \leq M$ for every $(t, x, v, 0)$. Then ν and σ in Hypothesis (H) can be chosen independently of R. Suppose $M \geq \nu$; set $\delta = \min\{\sigma, \frac{M}{L}\}$ and $C = \frac{1}{\nu} + \sqrt{\frac{1}{\nu^2} + \frac{1}{4M^2}}$. Fix $(t_0, x_0) \in \boldsymbol{R} \times (\mathcal{T}_\delta \setminus \mathcal{T})$ and inductively define a sequence (t_j, x_j) by setting $x_1 = x_0$, $t_j = \frac{\nu}{4M^2} d(x_j)$,$\tau_j = \sum_{k=0}^{j-1} t_k$ and $x_{j+1} = x_{\tau_j}(v_{\tau_j, x_j}, 0)(t_j)$ for $j \geq 1$, where v_{τ_j, x_j} is given by (H). With the same arguments exploited in [18], one can prove that $d_{\mathcal{T}}(x_{j+1}) \leq K^j d_{\mathcal{T}}(x_0)$, for $j \geq 0$, where $K = \sqrt{1 - (\frac{1}{2M})^2}$ and $\bar{t} = \sum_{k=1}^{+\infty} t_k \leq C d_{\mathcal{T}}(x_0)$. Therefore the sequence x_j converges to a point $\bar{x} \in \mathcal{T}$. Define the control

$$\overline{v}(t) = v_{\tau_j, x_j} \quad \text{for} \quad \tau_j \leq t \leq \tau_{j+1}.$$

We have $x_{t_0, x_0}(\overline{v}, 0)(\bar{t}) = \bar{x}$ and therefore $T(t_0, x_0, K) \leq C d_{\mathcal{T}}(x_0)$.

Next we have to prove (4.6) for general f and $\mathcal{T}$. For any $R > 0$ define $\mathcal{T}_{(R)} = \mathcal{T} \cap \mathcal{B}_\backslash[\prime, \in\mathcal{R}]$, $L_R = L_{B_n[0,2R]}$, $M_R = \max\{|f(t, x, v, 0)| : (t, x, v) \in B_{1+n}[0, 2R] \times V\}$, and

$$f_R(t, x, v, 0) = \begin{cases} f(t, x, v, 0) & \text{if} \quad |f(t, x, v, 0)| \leq M_R \\ M_R \frac{f(t, x, v, 0)}{|f(t, x, v, 0)|} & \text{if} \quad |f(t, x, v, 0)| > M_R. \end{cases}$$

Consider the control system obtained by substituting f and $\mathcal{T}$ with f_R and $\mathcal{T}_{(R)}$ and denote by T_R the corresponding minimum time function. By the first part of the proof, there exist $\delta_R = \min\{\sigma_R, \frac{M_R}{L_R}\}$ and C_R such that

$$T_R(t,x,K) \le C_R d_{\mathcal{T}}(x) \quad \text{for every} \quad (t,x) \in [-2R, 2R] \times (\mathcal{T}_{\delta_{\mathcal{R}}} \cap \mathcal{B}_{\setminus}[\prime, \in\mathcal{R}]).$$

Choose $\delta_R C_R M_R < R$. For such a choice of δ_R, we may assume that for ϵ small enough the ϵ-optimal trajectories starting at points $(t,x) \in [-R,R] \times (\mathcal{T}_{\delta_{\mathcal{R}}} \cap \mathcal{B}_{\setminus}[\prime, \mathcal{R}])$, that is $x_{t,x}(v,0)(t)$ with $\theta_{t,x}(v,0) \le T_R(t,x,K)+\epsilon$, remain inside $B_n[0,2R]$ and $t + T_R(t,x,K) + \epsilon \le 2R$. Therefore these trajectories are admissible for the original control system and we can deduce that $T(t,x,K) \le T_R(t,x,K) \le C_R d_{\mathcal{T}}(x)$.

Finally, to prove the claim about the reachable set $\mathcal{R}[K]$ observe that if $(\bar{t},\bar{x})$ belongs to $\mathcal{R}[K]$ and is sufficiently close to $\boldsymbol{R}\times\mathcal{T}$ then (4.6) implies that there exists a ball of suitable radius around $(\bar{t},\bar{x})$ on which $T(\cdot,\cdot,K)$ is bounded, that is $\mathcal{R}[K]$ contains a neighborhood of $\boldsymbol{R}\times\mathcal{T}$.

□

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