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Harmonic Forcing for Linear Distributed Parameter Systems*

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Abstract

This paper is concerned with providing simple conditions guaranteeing that a single input single output (SISO) linear distributed parameter system driven by a harmonic input will, for a suitable initial condition, possess a nontrivial periodic output with the same period as the input. This question is related to the problem of output regulation in the case of periodic tracking in which a system is driven by the output of a neutrally stable exosystem (such as a harmonic oscillator) and the objective is to design a feedback law that will force the output of the system to track the output of the exosystem.

1 Introduction

In this paper we consider a special class of Single Input Single Output (SISO) linear distributed parameter control systems in the form

$$\dot{z} = Az + bu, \tag{1.1}$$

$$z(0) = z_0, \tag{1.2}$$

$$y = cz \tag{1.3}$$

where A is the infinitesimal generator of a C_0 semigroup in a Hilbert space Z and $b \in \mathcal{L}(\mathbb{R}, Z)$, $c \in \mathcal{L}(Z, \mathbb{R})$. Here $\mathcal{L}(X, Y)$ denotes the space of bounded operators from X to Y.

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We assume that the input u is given, in feedback form, as the output of a harmonic oscillator with frequency α:

$$\begin{aligned}
&\dot{w} = Sw, \quad S = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \\
&w(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
&u = \Gamma w,
\end{aligned} \tag{1.4}$$

where Γ is a given 1×2 matrix: $\Gamma = [\gamma_1, \gamma_2]$. Thus u represents a periodic function of period $T = 2\pi/\alpha$ as a linear combination of $\sin(\alpha t)$ and $\cos(\alpha t)$, namely,

$$u(t) = \gamma_1 \sin(\alpha t) + \gamma_2 \cos(\alpha t). \tag{1.5}$$

Problem 1.1 *Suppose that we are given the input u in (1.5), find an initial condition z_0 in (1.2) so that the output y in (1.3) is a nontrivial periodic function of period $T = 2\pi/\alpha$.*

It is well known in finite dimensional linear control theory that if a system is driven by a periodic input for which the complex frequency $i\alpha$ is a transmission zero of the system, then the output of the system is zero for all time. Therefore, we should also state the following more general problem.

Problem 1.2 *Find conditions on (A, b, c) guaranteeing there is a nontrivial periodic output with the desired period for all α and arbitrary γ_1, γ_2 with $\gamma_1^2 + \gamma_2^2 \neq 0$, i.e., that the system will support a periodic output of arbitrary period.*

A similar scenario arises for the state feedback regulator problem in which one is interested in designing a feedback law to drive the closed loop system in order to have its output track a given periodic reference signal. Note that in the above case the input u is a periodic function of period $T = 2\pi/\alpha$. The questions posed in this paper are intermediate results in that our objective is to provide a simple test to determine whether it is even possible for the system to support a periodic output with the desired period T. For that matter we would also like to give criteria that would ensure that the system could support a periodic motion of arbitrary period.

Assume for the moment that (1.1) is a finite dimensional system with the state space $Z = \mathbb{R}^n$ and denote by $\{\lambda_j\}_{j=1}^n$ the spectrum of A (eigenvalues listed by multiplicity). In order that the solution to (1.1) be periodic we would at least need that $z(T) = z(0)$. By the variation of parameters formula we have

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-\tau)} b u(\tau)\, d\tau. \tag{1.6}$$

In order that $z(\cdot)$ satisfy $z(T) = z(0) = z_0$ we need

$$\left(I - e^{AT}\right) z_0 = \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau. \tag{1.7}$$

This equation is solvable for z_0 if either 1 is in the resolvent set of e^{AT} or more generally if the right hand side is in the range of $\left(I - e^{AT}\right)$. Our objective is to give conditions on the original system that will guarantee solvability. For finite dimensional systems the spectral mapping theorem (cf. [3], page 312, Theorem 6), which is valid for very general functions defined on the spectrum of A, gives

$$\sigma\left(e^{AT}\right) = e^{\sigma(A)T} \tag{1.8}$$

so that $\left(I - e^{AT}\right)$ is invertible if

$$1 \notin \sigma\left(e^{AT}\right).$$

This is true if

$$\lambda_j T \neq 2k\pi i, \quad \text{for } j = 1, \cdots n, \quad k = 0, \pm 1, \pm 2, \ \cdots ,$$

or, in other words, if

$$\lambda_j \neq k\alpha i, \quad j = 1, \cdots n, \quad k = 0, \pm 1, \pm 2, \ \cdots .$$

In the typical applications of interest to the authors, the spectrum of the unbounded operator A consists of a discrete set of eigenvalues of finite multiplicity whose real parts tend to minus infinity. Indeed, for λ in the resolvent set of A, the resolvent operator $(\lambda I - A)^{-1}$ is compact. In this case, due to the fact that the function $e^{\lambda t}$ has an essential singularity at $\lambda = \infty$, the the spectral mapping theorem found in standard texts on functional analysis (cf, [1, 6]) does not apply since it requires the functions to be analytic in a neighborhood of the spectrum. There are of course many special cases. For example if A is normal (e.g., if A is even selfadjoint) the the spectral mapping theorem holds for any continuous function f defined in a neighborhood of the spectrum, i.e., $f(\sigma(A)) = \sigma(f(A))$. In general, if A is the infinitesimal generator of a C_0 semigroup then the most one can say is that $e^{\sigma(A)} \subset \sigma(e^{AT})$ (cf., [4] (pages 45-48)). For the point spectrum we have

$$e^{\sigma_p(A)T} \subset \sigma_p(e^{AT}) \subset e^{\sigma_p(A)T} \cup \{0\}.$$

This holds, for example, if A is a discrete spectral operator whose eigenfunctions form a Riesz basis, which is usually the case for systems governed by partial differential equations on bounded domains.

Assumption 1.1 *In this paper we will avoid the various technical difficulties and make the assumption that A is a discrete Riesz spectral operator with simple eigenvalues (multiplicity one) $\{\lambda_j\}_{j=1}^{\infty}$ and eigenvectors $\{\psi_j\}_{j=1}^{\infty}$. These eigenvectors form a Riesz basis in Z (i.e., a linear isomorphic image of an orthonormal basis). In this case the adjoint A^* is also a discrete Riesz spectral operator whose eigenvectors $\{\psi_j^*\}_{j=1}^{\infty}$ form a biorthogonal Riesz basis, i.e.,*

$$\langle \psi_j, \psi_k^* \rangle = \delta_{jk}.$$

For such operators we have a functional calculus much like the case of self-adjoint operators. Namely, we have

$$\sum_{j=1}^{\infty} \langle \phi, \psi_j^* \rangle \psi_j = \phi, \text{ for all } \phi \in Z \tag{1.9}$$

$$\sum_{j=1}^{\infty} \lambda_j \langle \phi, \psi_j^* \rangle \psi_j = A\phi, \text{ for all } \phi \in \mathcal{D}(A) \tag{1.10}$$

$$\sum_{j=1}^{\infty} e^{\lambda_j t} \langle \phi, \psi_j^* \rangle \psi_j = e^{At}\phi, \text{ for all } \phi \in Z \tag{1.11}$$

$$\sum_{j=1}^{\infty} \frac{\langle \phi, \psi_j^* \rangle}{(1 - e^{\lambda_j T})} \psi_j = (I - e^{AT})^{-1}\phi, \text{ for all } \phi \in Z \tag{1.12}$$

$$\sum_{j=1}^{\infty} \frac{\langle \phi, \psi_j^* \rangle}{(\lambda_j - \lambda)} \psi_j = (A - \lambda I)^{-1}\phi, \text{ for all } \phi \in Z \tag{1.13}$$

Note that due to our assumption that b and c are bounded rank one operators, we have a well defined transfer function given by

$$g(s) = c(sI - A)^{-1} b.$$

This is a complex valued function of the complex variable s which is analytic on the resolvent set of A. Furthermore the singularities of $g(s)$ occur at the eigenvalues of A and hence have finite multiplicity, i.e., they are poles in the terminology of analytic function theory. For all examples that we have in mind, this is actually a meromorphic function so that all the zeros are also isolated and of finite multiplicity and certainly have no finite accumulation points.

Assumption 1.2 *A natural assumption on our system is that the transfer function is real, i.e.,*

$$g(\overline{s}) = \overline{g(s)}. \tag{1.14}$$

For systems governed by differential equations with real coefficients this condition is automatic.

Definition 1.1 *A complex number s_0 is a transmission zero if $g(s_0) = 0$.*

Assumption 1.3 *Our final assumption is that there are no pole zero cancellations. That is, we assume that if s_0 is a transmission zero, then $s_0 \in \rho(A)$, the resolvent set of A.*

2 Main Results

Theorem 2.1 *Let the operator A in (1.1) be a discrete Riesz spectral operator with $\sigma(A) = \{\lambda_j\}_{j=1}^{\infty}$, the input u is given by (1.5) with $\gamma_1^2 + \gamma_2^2 \neq 0$ and let (A, b, c) satisfy Assumptions 1.1, 1.2 and 1.3. Then we have the following results.*

1. *The solution z to (1.1) is periodic with period $T = 2\pi/\alpha$ provided*

$$\textit{distance}(\sigma(A), \{k\alpha i \mid k = 0, \pm 1, \pm 2, \cdots\}) > 0. \tag{2.1}$$

   Furthermore, the system supports all positive periods T (i.e., we can find a periodic solution for all possible frequencies α) if

$$\textit{distance}(\sigma(A), \mathbb{C}^0) > 0$$

   where $\mathbb{C}^0 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 0\}$ denotes the imaginary axis.

2. *In this case, there is a nontrivial periodic output y if and only if $i\alpha$ is not a transmission zero, i.e., $g(i\alpha) \neq 0$.*

3. *Finally, let us denote the amplitude of the periodic input u by*

$$M_u \equiv \sup_{t \in [0,T]} |u(t)| = \sqrt{\gamma_1^2 + \gamma_2^2}.$$

   Then the amplitude of the output y is a linear function of the amplitude of the input u. In particular, the output can be written in the forms

$$y(t) = \big[\operatorname{Re} g(i\alpha)\big] u(t) + \frac{1}{\alpha}\big[\operatorname{Im} g(i\alpha)\big] \frac{du}{dt}(t) \tag{2.2}$$

$$= M_u |g(i\alpha)| \big[\tilde{\gamma_1} \sin(\alpha t) + \tilde{\gamma_2} \cos(\alpha t)\big]$$

$$= M_u |g(i\alpha)| \sin(\alpha t + \phi) \tag{2.3}$$

   where $\tilde{\gamma_1}^2 + \tilde{\gamma_2}^2 = 1$ and we can easily write explicit formulas for $\tilde{\gamma_1}$, $\tilde{\gamma_2}$ and ϕ in terms of γ_1, γ_2 and $g(i\alpha)$. Thus the amplitude M_y of y can be written as

$$M_y \equiv \sup_{t \in [0,T]} |y(t)| = M_u |g(i\alpha)|.$$

Proof of 1:

Under the assumptions imposed on the operator A in Assumption 1.1, the solution of (1.1)-(1.2) is given by the variation of parameters formula

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-\tau)} b u(\tau)\, d\tau. \tag{2.4}$$

Therefore, just as in the finite dimensional case, in order that $z(\cdot)$ satisfy $z(T) = z(0) = z_0$ we need

$$\left(I - e^{AT}\right) z_0 = \int_0^T e^{A(T-\tau)} b u(\tau)\, d\tau. \tag{2.5}$$

From the representation in (1.12) for $(I - e^{AT})^{-1}$, we see that $(I - e^{AT})^{-1}$ is bounded provided the numbers $(1 - e^{\lambda_j T})$ are bounded away from zero. Under the assumption (2.1) we can thus solve (2.5) to obtain an initial condition

$$z_0 = \left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-\tau)} b u(\tau)\, d\tau \tag{2.6}$$

for which $z(T) = z(0)$. We now show that for any T periodic input u and the particular initial condition given in (2.6) the resulting solution to (1.1) given by the variation of parameter formula (cf., (2.4)) is T periodic, i.e., $z(t+T) = z(t)$ for all t. Namely, using the fact that

$$e^{AT} \left(I - e^{AT}\right)^{-1} \phi = \left(I - e^{AT}\right)^{-1} \phi - \phi$$

for all ϕ, we have for the initial condition z_0 in (2.6),

$$
\begin{aligned}
z(t+T) = e^{A(T+t)} & \left[\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau \right] \\
& + \int_0^{T+t} e^{A(T+t-\tau)} bu(\tau)\, d\tau \\
= \quad & e^{At} \left[e^{AT} \left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau \right] \\
& + \int_0^{T+t} e^{A(T+t-\tau)} bu(\tau)\, d\tau \\
= \quad & e^{At} \left[\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau \right] \\
& - e^{At} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau + \int_0^{T+t} e^{A(T+t-\tau)} bu(\tau)\, d\tau \\
= \quad & e^{At} z_0 - e^{At} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau \\
& + e^{At} \int_0^T e^{A(T-\tau)} bu(\tau)\, d\tau + \int_T^{T+t} e^{A(T+t-\tau)} bu(\tau)\, d\tau \\
= \quad & e^{At} z_0 + \int_0^t e^{A(t-\tau)} bu(\tau)\, d\tau \\
= \quad & z(t).n
\end{aligned}
$$

□

Proof of 2 and 3: In order to prove part 2 of Theorem 2.1, that the system (1.1) supports a nontrivial periodic output y with period T if and only if $g(i\alpha) \neq 0$, we must consider

$$y(t) = cz(t).$$

Therefore we need an explicit representation for the solution z.

First let us use the functional calculus to simplify the expression for the initial condition z_0 and the solution z. From (2.6) and (1.12) the initial condition can be written as

$$z_0 = \sum_{j=1}^{\infty} \frac{e^{\lambda_j T}}{(1 - e^{\lambda_j T})} \left(\int_0^T e^{-\lambda_j \tau} u(\tau)\, d\tau \right) \langle b, \psi_j^* \rangle \psi_j.$$

Now from the explicit form of u given in (1.5) we can readily compute

$$\int_0^T e^{-\lambda_j \tau} \sin(\alpha\tau)\, d\tau = \frac{\alpha \left(1 - e^{-\lambda_j T}\right)}{(\lambda_j^2 + \alpha^2)},$$

and

$$\int_0^T e^{-\lambda_j \tau} \cos(\alpha\tau)\, d\tau = \frac{\lambda_j \left(1 - e^{-\lambda_j T}\right)}{(\lambda_j^2 + \alpha^2)},$$

to obtain

$$\begin{aligned} z_0 &= \sum_{j=1}^{\infty} \frac{e^{\lambda_j T}}{(1 - e^{\lambda_j T})} \left(\int_0^T e^{-\lambda_j \tau} u(\tau)\, d\tau \right) \langle b, \psi_j^* \rangle \psi_j \\ &= \sum_{j=1}^{\infty} \frac{e^{\lambda_j T}}{(1 - e^{\lambda_j T})} \left[\gamma_1 \frac{\alpha \left(1 - e^{-\lambda_j T}\right)}{(\lambda_j^2 + \alpha^2)} + \gamma_2 \frac{\lambda_j \left(1 - e^{-\lambda_j T}\right)}{(\lambda_j^2 + \alpha^2)} \right] \langle b, \psi_j^* \rangle \psi_j \\ &= \sum_{j=1}^{\infty} \left[\frac{-\gamma_1 \alpha - \gamma_2 \lambda_j}{(\lambda_j^2 + \alpha^2)} \right] \langle b, \psi_j^* \rangle \psi_j \\ &= -\big(\gamma_1 \alpha + \gamma_2 A\big)(A^2 + \alpha^2)^{-1} b. \end{aligned} \tag{2.7}$$

In the same way, using the formulas

$$\int_0^t e^{-\lambda_j \tau} \sin(\alpha\tau)\, d\tau = \frac{(-\alpha \cos(\alpha t) - \lambda_j \sin(\alpha t))\, e^{-\lambda_j t}}{(\lambda_j^2 + \alpha^2)} + \frac{\alpha}{(\lambda_j^2 + \alpha^2)},$$

and

$$\int_0^t e^{-\lambda_j \tau} \cos(\alpha\tau)\, d\tau = \frac{(-\lambda_j \cos(\alpha t) + \alpha \sin(\alpha t))\, e^{-\lambda_j t}}{(\lambda_j^2 + \alpha^2)} + \frac{\lambda_j}{(\lambda_j^2 + \alpha^2)},$$

we easily obtain the following explicit representation for the solution z.

$$
\begin{aligned}
z(t) &= e^{At}z_0 + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau \\
&= e^{At}z_0 + \sum_{j=1}^{\infty} e^{\lambda_j t}\left[\int_0^t e^{-\lambda_j \tau}\big(\gamma_1 \sin(\alpha\tau) + \gamma_2 \cos(\alpha\tau)\big)\,d\tau\right]\langle b, \psi_j^*\rangle \psi_j \\
&= e^{At}z_0 + \sum_{j=1}^{\infty} e^{\lambda_j t}\left(\frac{(\gamma_1\alpha + \gamma_2\lambda_j)}{(\lambda_j^2 + \alpha^2)}\right)\langle b, \psi_j^*\rangle \psi_j \\
&\quad + \sin(\alpha t)\sum_{j=1}^{\infty}\left(\frac{(-\gamma_1\lambda_j + \gamma_2\alpha)}{(\lambda_j^2 + \alpha^2)}\right)\langle b, \psi_j^*\rangle \psi_j \\
&\quad + \cos(\alpha t)\sum_{j=1}^{\infty}\left(\frac{(-\gamma_1\alpha - \gamma_2\lambda_j)}{(\lambda_j^2 + \alpha^2)}\right)\langle b, \psi_j^*\rangle \psi_j \\
&= -e^{At}\big(\gamma_1\alpha + \gamma_2 A\big)(A^2 + \alpha^2)^{-1}b + e^{At}\big(\gamma_1\alpha + \gamma_2 A\big)(A^2 + \alpha^2)^{-1}b \\
&\quad + \sin(\alpha t)\big(-\gamma_1 A + \gamma_2\alpha\big)(A^2 + \alpha^2)^{-1}b \\
&\quad + \cos(\alpha t)\big(-\gamma_1 \alpha - \gamma_2 A\big)(A^2 + \alpha^2)^{-1}b \\
&= \Big[\sin(\alpha t)\big(-\gamma_1 A + \gamma_2\alpha\big) + \cos(\alpha t)\big(-\gamma_1\alpha - \gamma_2 A\big)\Big](A^2 + \alpha^2)^{-1}b. \qquad (2.8)
\end{aligned}
$$

Applying c to (2.8) we obtain

$$
y(t) = c\Big[\sin(\alpha t)\big(-\gamma_1 A + \gamma_2\alpha\big) + \cos(\alpha t)\big(-\gamma_1\alpha - \gamma_2 A\big)\Big](A^2 + \alpha^2)^{-1}b. \qquad (2.9)
$$

Our next objective is to interpret the formula for y given in (2.9) in terms of the transfer function $g(s) = c(sI - A)^{-1}b$. To this end recall the resolvent identity for $\lambda, \mu \in \rho(A)$,

$$
(\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} = (\lambda - A)^{-1} - (\mu - A)^{-1}.
$$

With $\mu = i\alpha$ and $\lambda = -i\alpha$ we have

$$
(\alpha^2 + A^2)^{-1} = (i\alpha - A)^{-1}(-i\alpha - A)^{-1} = \frac{1}{2i\alpha}\left[(-i\alpha - A)^{-1} - (i\alpha - A)^{-1}\right] \qquad (2.10)
$$

Also note that

$$
A(i\alpha - A)^{-1} = i\alpha(i\alpha - A)^{-1} - I,\ A(-i\alpha - A)^{-1} = -i\alpha(-i\alpha - A)^{-1} - I, \qquad (2.11)
$$

which implies

$$A(\alpha^2 + A^2)^{-1} = \frac{-1}{2}\big[(-i\alpha - A)^{-1} + (i\alpha - A)^{-1}\big] \tag{2.12}$$

Now using the fact that the transfer function is real (cf, (1.14)), we have from (2.10)

$$\begin{aligned} \alpha c(\alpha^2 + A^2)^{-1} b &= \frac{\alpha}{2i\alpha}\big[c(-i\alpha - A)^{-1} b - c(i\alpha - A)^{-1} b\big] \\ &= \frac{\alpha}{2i\alpha}[g(-i\alpha) - g(i\alpha)] = -\frac{1}{2i}[g(i\alpha) - \overline{g(i\alpha)}] \\ &= -\operatorname{Im} g(i\alpha), \end{aligned} \tag{2.13}$$

and from (2.12)

$$\begin{aligned} cA(\alpha^2 + A^2)^{-1} b &= -\frac{1}{2}\big[c(-i\alpha - A)^{-1} b + c(-\alpha - A)^{-1} b\big] \\ &= -\frac{1}{2}[g(-i\alpha) + g(i\alpha)] = -\frac{1}{2}[g(i\alpha) + \overline{g(i\alpha)}] \\ &= -\operatorname{Re} g(i\alpha). \end{aligned} \tag{2.14}$$

The formulas (2.13) and (2.14) allow us to rewrite (2.9) as

$$\begin{aligned} y(t) &= \sin(\alpha t)\big[\gamma_1 \operatorname{Re} g(i\alpha) - \gamma_2 \operatorname{Im} g(i\alpha)\big] \\ &\quad + \cos(\alpha t)\big[\gamma_1 \operatorname{Im} g(i\alpha) + \gamma_2 \operatorname{Re} g(i\alpha)\big] \qquad (2.15) \\ &= \big[\gamma_1 \sin(\alpha t) + \gamma_2 \cos(\alpha t)\big] \operatorname{Re} g(i\alpha) \\ &\quad + \big[\gamma_1 \cos(\alpha t) - \gamma_2 \sin(\alpha t)\big] \operatorname{Im} g(i\alpha) \\ &= u(t) \operatorname{Re} g(i\alpha) + \frac{1}{\alpha}\frac{du}{dt}(t) \operatorname{Im} g(i\alpha). \qquad (2.16) \end{aligned}$$

We can now answer the question of when y is nontrivial. A straightforward calculation show that

$$\begin{aligned} 0 = y(t) \quad &= \quad \sin(\alpha t)\big[\gamma_1 \operatorname{Re} g(i\alpha) - \gamma_2 \operatorname{Im} g(i\alpha)\big] \\ &\quad + \cos(\alpha t)\big[\gamma_1 \operatorname{Im} g(i\alpha) + \gamma_2 \operatorname{Re} g(i\alpha)\big], \end{aligned}$$

for all t if and only if there are γ_1, γ_2, not both zero, such that

$$\begin{aligned} \gamma_1 \operatorname{Re} g(i\alpha) - \gamma_2 \operatorname{Im} g(i\alpha) = 0, \\ \gamma_1 \operatorname{Im} g(i\alpha) + \gamma_2 \operatorname{Re} g(i\alpha) = 0. \end{aligned}$$

The determinant of the coefficient matrix for this system is

$$|g(i\alpha)|^2 = (\operatorname{Re} g(i\alpha))^2 + (\operatorname{Im} g(i\alpha))^2.$$

From this we see that y represents a nontrivial periodic motion if and only if $g(i\alpha) \neq 0$, i.e., if and only if $i\alpha$ is not a transmission zero.

Finally, to prove part 3 of the Theorem in the formula (2.15) for y let

$$\delta_1 = \big(\gamma_1 \operatorname{Re} g(i\alpha) - \gamma_2 \operatorname{Im} g(i\alpha)\big),$$

and

$$\delta_2 = \big(\gamma_1 \operatorname{Im} g(i\alpha) + \gamma_2 \operatorname{Re} g(i\alpha)\big).$$

With this (2.15) can be written as

$$\begin{aligned} y(t) &= \left(\delta_1^2 + \delta_2^2\right) \left[\frac{\delta_1}{(\delta_1^2 + \delta_2^2)} \sin(\alpha t) + \frac{\delta_2}{(\delta_1^2 + \delta_2^2)} \cos(\alpha t)\right] \\ &= \left(\delta_1^2 + \delta_2^2\right) \sin(\alpha t + \phi) \end{aligned}$$

where

$$\sin(\phi) = \frac{\delta_2}{(\delta_1^2 + \delta_2^2)}, \quad \cos(\phi) = \frac{\delta_1}{(\delta_1^2 + \delta_2^2)}.$$

From this we easily see that the amplitude of y is given by $\sqrt{(\delta_1^2 + \delta_2)}$ which can be rewritten in terms of the amplitude of u and the magnitude of the transfer function. Namely, we have

$$\begin{aligned} M_y^2 = \left(\delta_1^2 + \delta_2^2\right) &= \big(\gamma_1 \operatorname{Re} g(i\alpha) - \gamma_2 \operatorname{Im} g(i\alpha)\big)^2 \\ &\quad + \big(\gamma_1 \operatorname{Im} g(i\alpha) + \gamma_2 \operatorname{Re} g(i\alpha)\big)^2 \\ &= \left(\gamma_1^2 + \gamma_2^2\right)\left(\operatorname{Re} g(i\alpha)^2 + \operatorname{Im} g(i\alpha)^2\right) \\ &= \left(\gamma_1^2 + \gamma_2^2\right) |g(i\alpha)|^2 = M_u^2 |g(i\alpha)|^2 \end{aligned}$$

and hence

$$M_y = M_u \, |g(i\alpha)|.$$

□

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