Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 2, 1998, pp. 1–8

Observability on Noncompact Symmetric Spaces*

Joseph A. Wolf†

1 The General Problem

Let X be a differentiable manifold and D a differential operator on X. Let $f(x:t)$ be a solution to the evolution equation

$$D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0 \qquad (x \in X,\ t \geqq 0). \tag{1.1}$$

Choose an "observation time" $t_0 > 0$. Our problem is to find points $\{x_0, x_1, \dots\} \subset X$ such that

(a) the values $f(x_i, t_0), 1 \leqq i \leqq n$, determine a reasonable approximation $b_n(x)$ to the initial data $b(x) = f(x:0)$,

(b) $\lim_{n\to\infty} b_n(x) = b(x)$ in some reasonable way, and

(c) we understand the speed of convergence well enough to know when to stop.

The "classical case" is the case in which X is a compact riemannian manifold and D is the (positive definite) Laplacian. Then (1.1) is the heat equation on X. In this paper we'll look at the special case where X is a riemannian symmetric space of noncompact type. Thus X is a noncompact riemannian manifold with a very large symmetry group G, harmonic analysis on X is understood in terms of the structure of G, and the operator D is G–invariant. The idea is to use some geometry and group structure to guide methods of observation, control and quadrature.

*Received November 5, 1996; received in final form June 26, 1997. Summary appeared in Volume 8, Number 2, 1998. This paper was presented at the Conference on Computation and Control V, Bozeman, Montana, August 1996. The paper was accepted for publication by special editors John Lund and Kenneth Bowers.

†Research partially supported by NSF Grant DMS 93 21285

Wallace and I had looked at this situation for compact X, specifically when X is a compact homogeneous or symmetric space G/K. See [12] for the observability and [13] for the speed of convergence.

At this point one should ask why we are looking at such complicated models. The reason is that a lot of special function theory and approximation theory, usually viewed analytically, can also be viewed geometrically. The point is much of special function theory is tied to group representation theory and the geometry of riemannian symmetric spaces. This is old news, but we mention it again to emphasize the fact that one can look to non–euclidean geometry, as well as euclidean geometry, as a guide to setting up mathematical models. Here we refer the reader to Helgason's books [8] and [9] for an introduction to geometry and analysis on symmetric spaces.

2 Review of Compact Case

We review the setting and indicate some of the results of [12] and [13].

Let $S = K/M$, compact riemannian homogeneous space. Thus K is a compact Lie group, S is a riemannian manifold, and K acts smoothly and transitively on S preserving the riemannian metric. Choose a base point $s_0 \in S$ and set $M = \{k \in K \mid k(s_0) = s_0\}$. Then S is in bijective correspondence $k(s_0) \leftrightarrow kM$ with the coset space K/M.

For example one might have $S = S^n$, unit sphere in $\mathbb{R}^{n+1}$ with induced riemannian metric of constant curvature $+1$, with $s_0 = {}^t(0, \dots, 0, 1)$, column vector, with $K = SO(n+1)$ rotation group, and with $M = SO(n)$.

Let D be a closed K–invariant differential operator on S. In the example of S^n, D could be any polynomial in the positive definite Laplace–Beltrami operator Δ. In any case consider the invariant evolution equation with initial data $b(s)$, given by

$$D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0,\ \ f(s:0) = b(s) \tag{2.1}$$

for $x \in X, t \geqq 0$. Invariance and the Peter–Weyl Theorem show that D is a normal operator on $L^2(S)$ and that

$$L^2(S) = \sum_{j=0}^{\infty} A(\lambda_j) \text{ where } A(\lambda_j) = \lambda_j\text{–eigenspace of } D. \tag{2.2}$$

Here D has discrete spectrum, again by K–invariance of D and the Peter–Weyl Theorem. In the special case $S = S^n$ and $D = \Delta$ one has $\lambda_j = \frac{(n-1)j+j^2}{2n-2}$ and $\dim A(\lambda_j) = \frac{n-1+2j}{n-1} \prod_{k=1}^{n-2} \frac{k+j}{k}$. In general, choose

$$\{\phi_{j,1}, \dots, \phi_{j,d_j}\} : \text{ orthonormal basis for } A(\lambda_j). \tag{2.3}$$

Then the general solution to (2.1) with L^2 initial data is

$$f(s:t) = \sum_{j=0}^{\infty} \sum_{i=1}^{d_j} c_{j,i} e^{-t\lambda_j} \phi_{j,i}(s) \quad \text{for} \quad s \in S, t \geqq 0 \text{ with } \sum |c_{j,i}|^2 < \infty. \tag{2.4}$$

The observability problem is to recover the coefficients $c_{j,i}$, $1 \leqq j \leqq r$, from the appropriate number $d_0 + \cdots + d_r = n_r$ of point evaluations

$$f_r(s:t_0) = \sum_{j=0}^{r} \sum_{i=1}^{j} c_{j,i} e^{-t_0\lambda_j} \phi_{j,i}(s) \tag{2.5}$$

of the truncated sums for $f(s:t)$. The acuity problem is to find the speed of convergence of the $\{f_r\} \to f$. This is done in [12] and [13].

3 Noncompact Symmetric Spaces

We now consider a situation in which the manifold and the group are noncompact, the case where X is a riemannian symmetric space of noncompact type, G is the largest connected group of isometries, and D is a G–invariant differential operator on X. Here the analysis combines that of the compact case described in Section 2 above, with somewhat more classical methods for the euclidean case.

We recall some of the basic structural results on X and G. First, G is a connected semisimple Lie group with center reduced to $\{1\}$. The isotropy subgroups of G on X are just the maximal compact subgroups. Choose a base point $x_0 \in X$, or, equivalently, the maximal compact subgroup $K = \{g \in G \mid g(x_0) = x_0\}$ in G.

The first example, real hyperbolic space, is the open unit ball $X = \{x \in \mathbb{R}^n \mid ||x|| < 1\}$. The connected special orthogonal group $\widetilde{G} = SO(n,1)$ acts on X by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : x \to (ax+b)(cx+d)^{-1} \tag{3.1}$$

where a is $n \times n$, b and x are $n \times 1$, c is $1 \times n$ and d is 1×1. Here we take x_0 to be the zero vector, so $K = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\} \cong SO(n)$.

Write $\mathfrak{g}$ and $\mathfrak{k}$ for the respective Lie algebras (algebras of infinitesimal generators) of G and K. Conjugation $g \mapsto s_{x_0} g s_{x_0}^{-1}$ by the symmetry s_{x_0} of X at x_0, is an involutive automorphism θ of G with fixed point set K. We also write θ for its differential. Now the decomposition of $\mathfrak{g}$ into (± 1)–eigenspaces of θ is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and let A denote the corresponding analytic subgroup of G. Define $M = Z_K(\mathfrak{a})$ and $M' = N_K(\mathfrak{a})$, centralizer and normalizer of $\mathfrak{a}$ (or, equivalently, of A) in

K. We write $\mathfrak{m}$ for their Lie algebra. The quotient $W = M'/M$ is the *Weyl group*, a finite group that acts by conjugation on $\mathfrak{a}$ and, by duality, on $\mathfrak{a}^*$. We will need it below. The quotient K/M, which we also need below, is the *Furstenberg boundary* or *minimal boundary* of X, ideal boundary on which bounded harmonic functions take their maximal values [5]. We will use it for an extension of the idea of polar coordinates and only tangentially for its potential–theoretic properties.

Some basic facts: $A(x_0)$ is a maximal flat totally geodesic submanifold of X and $G = KAK$. Thus $X = KA(x_0)$ and we have surjective maps

$$(K/M') \times A(x_0) \to X \text{ and } (K/M) \times (M'\backslash A(x_0)) \to X \tag{3.2}$$

defined by $(kM, a(x_0)) \mapsto ka(x_0)$.

Let's look at this when X is the real hyperbolic n–space $\mathbb{H}^n$. We view $\mathbb{H}^n$ in Poincaré's model, as the open unit ball in real euclidean n–space $\mathbb{R}^n$. Its geodesics are the circular arcs or straight line segments inside the unit ball that meet the boundary sphere orthogonally. We parameterize $\mathbb{H}^n$ using polar coordinates (t, ϕ) where t is radial distance (in the hyperbolic metric) from the base point $x_0 = 0 \in \mathbb{R}^n$ and ϕ is the coordinate on the unit sphere in the tangent space at x_0. Thus $k_\phi a_t(x_0)$ has coordinates (t, ϕ) where $a_t = \left(\begin{smallmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{smallmatrix}\right) \in G$ and k_ϕ is any $\left(\begin{smallmatrix} k'_\phi & 0 \\ 0 & 1 \end{smallmatrix}\right) \in K$ such that $\phi = k'_\phi \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is the first column of $k'_\phi \in SO(n)$. In other words $\mathfrak{a} = \left\{ \left(\begin{smallmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{smallmatrix}\right) \middle| \, t \text{ real} \right\}$, so

$$A = \left\{ \left(\begin{smallmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{smallmatrix}\right) \middle| \, t \text{ real} \right\} \text{ and } A(x_0) = \left\{ \begin{pmatrix} \tanh t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \middle| \, t \text{ real} \right\},$$

and any element $k = \left(\begin{smallmatrix} k' & 0 \\ 0 & 1 \end{smallmatrix}\right), k' \in SO(n)$, acts linearly as k on the ambient $\mathbb{R}^n$. Note that the range of $\tanh t$ is the open interval $(-1, +1)$. Now

$$M = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \middle| \, m \in SO(n-1) \right\},$$

$$M' = \left\{ \left(\begin{smallmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon m & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \middle| \, m \in SO(n-1) \text{ and } \epsilon = \pm 1 \right\},$$

and

$$M'\backslash A(x_0) = \left\{ \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix} \middle| \, 0 \leqq u < 1 \right\}.$$

The *adjoint representation* of $\mathfrak{g}$ is the Lie algebra representation of $\mathfrak{g}$ on itself defined by $\mathrm{ad}(\xi) : \eta \mapsto [\xi, \eta]$. Here $[\cdot, \cdot]$ is the Lie algebra product. If $\mathfrak{g}$ is a matrix Lie algebra it is given by the commutator, $[\xi, \eta] = \xi\eta - \eta\xi$. In any case $\mathrm{ad}(\mathfrak{a})$ is simultaneously diagonalizable. The joint eigenvalues are linear functionals $\alpha \in \mathfrak{a}^*$. The nonzero ones are called the $\mathfrak{a}$–*roots* or *restricted* *roots*. Write $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ for the set of $\mathfrak{a}$–roots. If $\alpha \in \Sigma$ the corresponding *root space* is the joint eigenspace $\mathfrak{g}_\alpha = \{\eta \in \mathfrak{g} \mid [\xi, \eta] = \alpha(\xi)\eta \text{ for all } \xi \in \mathfrak{a}\}$.

If $\alpha \in \Sigma$ then $-\alpha \in \Sigma$ as well. A *positive* $\mathfrak{a}$*–root system* is a subset $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a}) \subset \Sigma$ such that (i) $\Sigma = \Sigma^+ \cup (-\Sigma^+)$, disjoint, and (ii) if $\alpha, \beta \in \Sigma^+$ and $\alpha + \beta \in \Sigma$ then $\alpha + \beta \in \Sigma^+$. The Weyl group W acts simply transitively on the set of all positive $\mathfrak{a}$–root systems. Fix a choice of Σ^+. Let ρ denote half the sum of the positive roots, with multiplicity, in other words $\rho = \frac{1}{2}\Sigma_{\alpha \in \Sigma^+}(\dim \mathfrak{g}_\alpha)\alpha$.

Every $\lambda \in \mathfrak{a}^*$ defines a positive definite spherical function $\phi_\lambda : X \to \mathbb{C}$, by the equation

$$\phi_\lambda(kax_0) = e^{-(i\lambda+\rho)(\xi)}, \quad k \in K, a \in A, \xi = \log a \in \mathfrak{a}. \tag{3.3}$$

Here $\phi_{w\lambda} = \phi_\lambda$ for all $w \in W$. The *spherical transform* on X is the map from functions $f : X \to \mathbb{C}$ to functions $\widehat{f} : (\mathfrak{a}^*/W) \times (K/M) \to \mathbb{C}$, given by

$$\widehat{f}(\lambda, kM) = \int_G f(g)\phi_\lambda(g^{-1}k)dg \tag{3.4}$$

whenever the integral converges. Fact: if $f \in C_c^\infty(X)$ then $\widehat{f} \in \mathcal{C}((\mathfrak{a}^*/W) \times (K/M))$, the space of rapidly decreasing C^∞ (Schwartz class) functions on $(\mathfrak{a}^*/W) \times (K/M)$. See (4.1) below.

If $\alpha \in \Sigma$ we write $m(\alpha)$ for its multiplicity, $m(\alpha) = \dim \mathfrak{g}_\alpha$. If α is a multiple of another $\mathfrak{a}$–root, say $\alpha = n\beta$, then $n = \pm 1$ or $n = \pm 2$. We write $m(\alpha/2)$ for $m(\beta)$ if $\alpha = 2\beta$, for 0 if $\alpha/2 \notin \Sigma$.

Write Σ_0 for the system of *indivisible roots*, that is, $\mathfrak{a}$–roots $\beta \in \Sigma$ such that $\frac{1}{2}\beta \notin \Sigma$. Denote $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$.

The Plancherel density on X is defined by the famous c–function of Harish–Chandra[1]. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ then

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\langle i\lambda, \alpha_0 \rangle}\Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))}$$

where $\alpha_0 = \frac{\alpha}{\langle \alpha, \alpha \rangle}$ and c_0 is the constant specified by $c(-\rho) = 1$. The Plancherel density is $|c(\cdot)|^{-2}$. It occurs in both the Plancherel Theorem and the Fourier Inversion Formula below. An example: if X is the real hyperbolic plane $\{x \in \mathbb{R}^2 \mid ||x|| < 1\}$ then $|c(\lambda)|^{-2}d\lambda = \frac{1}{2\pi}\lambda \tanh(\pi\lambda)d\lambda$.

Theorem 3.5 (Plancherel Theorem). *Let $f \in C_c^\infty(X)$ and define $\widehat{f}$ as in* (3.4). *Then*

[1]Harish–Chandra determined the c–function both for G complex and for G of real rank 1 in 1958 [7]. Then in 1960 Bhanu–Murthy determined $c(\lambda)$ for all but one of the classical simple groups G that are normal real forms ([1], [2]). Finally, in 1962 Gindikin and Karpelevič proved the general product formula for $c(\lambda)$ based on Harish–Chandra's rank 1 formulae [6]. See Helgason ([8] or [9]) for expositions.

(a) $\widehat{f} \in \mathcal{C}((\mathfrak{a}^*/W) \times (K/M))$,

(b) $\widehat{f} \in L^2(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda)\widehat{\otimes}L^2(K/M)$,

(c) $||\widehat{f}||_{L^2} = ||f||_{L^2}$, *and*

(d) *the norm–preserving linear map* $f \mapsto \widehat{f}$ *extends by continuity to an isometry* $L^2(X) \cong L^2(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda)\widehat{\otimes}L^2(K/M)$.

Theorem 3.6 (Fourier Inversion Theorem). *Let* $f \in C_c^\infty(X)$. *View* f *as a function on* G. *Then*

$$f(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*}\int_{K/M}\widehat{f}(\lambda, kM)\phi_{-\lambda}(g^{-1}k)|c(\lambda)|^{-2}d\lambda dk. \tag{3.7}$$

In Theorems 3.5 and 3.6 it is useful to note that (i) $c(-\lambda) = \overline{c(\lambda)}$, (ii) $|c(\lambda)| = |c(w\lambda)|$ for all $w \in W$, and (iii) there are integers $u, v > 0$ such that $|c(\lambda)|^{-1} \leqq u(1 + ||\lambda||)^v$. From (iii) we see that integration against $1/c(\lambda)$ is a tempered distribution on $\mathfrak{a}^*$.

4 The Product Decomposition

Theorems 3.5 and 3.6 play on a product decomposition for $L^2(X)$ and for $\mathcal{C}(X)$ which comes out of the analog (3.2) of polar coordinates on X. Here we indicate how that product decomposition carries some observability and approximation questions from K/M to X.

Let $\lambda \in \mathfrak{a}^*$. The positive definite spherical function ϕ_λ defines a Hilbert space $\mathcal{H}_\lambda$, and G acts on $\mathcal{H}_\lambda$ by a unitary representation π_λ . We can view the elements of $\mathcal{H}_\lambda$ as limits of linear combinations of G–translates of ϕ_λ and conclude $\mathcal{H}_\lambda \subset C^\infty(G)$. See [9]. Let $\mathcal{D}(X)$ denote the algebra of G–invariant differential operators on X. It is commutative, and (a.e. $\lambda \in \mathfrak{a}^*$) the $\mathcal{H}_\lambda$ are its joint eigenspaces on $C^\infty(G)$.

The Schwartz space version of the Plancherel Theorem 3.5 says that the Fourier–Plancherel–Harish-Chandra transform $f \mapsto \widehat{f}$ is an isomorphism

$$\mathcal{F} : \mathcal{C}(X) \cong \mathcal{C}(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda)\widehat{\otimes}\mathcal{C}(K/M) \tag{4.1}$$

of nuclear Fréchet spaces. Here $\mathcal{C}(K/M) = C^\infty(K/M)$ because K/M is compact, and $\mathcal{C}(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda)$ denotes the space of all C^∞ and W–invariant functions ψ on $\mathfrak{a}^*$ such that

$$p(\frac{\partial}{\partial\lambda})\psi \in L^2(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda) \tag{4.2}$$

for every W–invariant polynomial differential operator $p(\frac{\partial}{\partial\lambda})$ on $\mathfrak{a}^*$.

We now view $\mathcal{C}(X)$ as the space of $\mathcal{C}(\mathfrak{a}^*/W, |c(\lambda)|^{-2}d\lambda)$–valued C^∞ functions on K/M. Then the method of [12] and [13] specifies observability and acuity for $T \cdot f : K/M \to \mathbb{C}$ whenever (i) $f(x : t)$ solves the evolution equation (1.1) and (ii) T is a tempered distribution on $\mathfrak{a}^*/W$ relative to the measure $|c(\lambda)|^{-2}d\lambda$. Here we are, in effect, imposing a certain degree of uniformity in K/M for tempered distributions on X.

Observability and approximation by methods of harmonic analysis on X is now reduced to two separate issues. They are consideration of the compact space K/M, by the method of [12] and [13], and an appropriate consideration of the euclidean space (modulo a finite symmetry group) $\mathfrak{a}^*/W \cong A(x_0)/W$. The appropriate methods for the latter are not yet clear, though of course they should reflect the action of A on the maximal flat totally geodesic submanifold $A(x_0) \subset X$ as euclidean translations and also the symmetries from the Weyl group W. Certainly the Sinc–Galerkin methods described by Stenger (see [10] and [11]), and Bowers and Lund (see, for example, [3] and [4]) appear to represent the best approach here.

References

[1] T.S. Bhanu–Murthy. Plancherel's measure for the factor space **SL**(n,**R**)/**SO**(n, **R**), *Dokl. Akad. Nauk SSSR*, **133** (1960), 503–506.

[2] T.S. Bhanu–Murthy. The asymptotic behavior of zonal spherical functions on the Siegel upper half–plane, *Dokl. Akad. Nauk SSSR*, **135** (1960), 1027–1029.

[3] K.L. Bowers and J. Lund. Numerical solution of singular Poisson problems via the Sinc–Galerkin method, *SIAM J. Numerical Analysis*, **24** (1987), 36–51.

[4] K.L. Bowers and J. Lund. Sinc methods for quadrature and differential equations, *SIAM*, 1982.

[5] H. Furstenberg. A Poisson formula for semisimple Lie groups, *Annals of Math* **77** (1963), 335–386.

[6] S.G. Gindikin and F.I. Karpelevič. Plancherel measure of Riemannian symmetric spaces of non–positive curvature, *Dokl. Akad. Nauk SSSR*, **145** (1962), 252–255.

[7] Harish–Chandra. Spherical functions on a semi–simple Lie group I, *American J. Math.* **80** (1958), 241–310.

[8] S. Helgason. *Groups and Geometric Analysis.* New York: Academic Press, 1984.

[9] S. Helgason. *Geometric Analysis on Symmetric Spaces.* Providence, RI: American Mathematical Society, 1994.

[10] F. Stenger. Numerical methods based on Whittaker cardinal, or sinc functions, *SIAM Rev.* **23** (1981), 165–224.

[11] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions.* New York: Springer–Verlag, 1993.

[12] D.I. Wallace and J.A. Wolf. Observability of evolution equations for invariant differential operators, *Journal of Mathematical Systems, Estimation, and Control,* **1** (1991), 29–44.

[13] D.I. Wallace and J.A. Wolf. Acuity of observation for invariant evolution equations, *Computation and Control II (Proceedings, Bozeman, 1990)*, Birkhauser Boston, Progress in Systems and Control Theory, 11 (1991), 325–350.

Department of Mathematics, University of California, Berkeley, California 94720–3840

Communicated by John Lund