Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 2, 1998, pp. 1–17

Uniform Exponential Stability of Approximations in Linear Viscoelasticity*

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Abstract

In this paper, we first study the abstract setting of linear viscoelasticity and under reasonable conditions on the kernel we prove the exponential stability of the associated semigroup. Secondly, we study the general approximate schemes proposed by Banks and Burns, by Ito and Kappel, and by Fabiano and Ito, respectively. The uniformly exponential stability of semigroups associated with these approximate schemes is proved.

AMS Subject Classifications: 35B40, 45L10, 73H10, 93D20
Key words: viscoelasticity, approximation, uniformly exponential stability, semigroup

1 Introduction

We consider the preservation of the uniform exponential stability of the approximation to the following integro-differential equation

$$\ddot{u}(t) + A\left[\epsilon u(t) - \int_{-r}^{0} g(s)u(t+s)ds\right] = f(t). \tag{1.1}$$

Here A is a positive definite, self-adjoint unbounded operator on a Hilbert space H, and $f(t)$ is an H-valued function; ϵ is a positive constant (a stiffness coefficient in applications to linear viscoelasticity); and $g(s)$ is a "history kernel" which satisfies the following conditions,

(g1) $g(s) \in C^1[-r,0) \cap L^1(-r,0)$, $g(s) > 0$ on $[-r,0)$,

*Received 1995; received in final form October 31, 1996. Summary appeared in Volume 8, Number 2, 1998.
†Supported by NSFC 19071919.

(g2) $g'(s) \geq \delta g(s)$ on $[-r, 0)$ for some constant $\delta > 0$,

(g3) $\alpha = \epsilon - \int_{-r}^{0} g(s)ds > 0$.

Equation (1.1) can be written as an abstract evolution equation

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) + \mathcal{F}(t), \tag{1.2}$$

in a Hilbert space Z. The term $\mathcal{F}(t)$ may represent a control in the form of $\mathcal{B}u(t)$, where $u \in R^m$ is a control input and $\mathcal{B}$ is a linear operator from R^m to Z. The most common approach for the approximation of control problem (1.2) is to consider a sequence of finite dimensional control systems of the form

$$\frac{d}{dt}z^N(t) = \mathcal{A}^N z^N(t) + \mathcal{B}^N u(t), \tag{1.3}$$

in the finite dimensional space $Z^N \subset Z$. Generally, system (1.3) is derived by discretization techniques for the approximation of the solution of (1.2). A control design is then obtained for this finite dimensional control problem, and is used as an approximation to the desired control of the original infinite dimensional control problem. The convergence of the scheme and the preservation of the uniform exponential stability for the uncontrolled system ($u = 0$ in (1.3)) are two most important conditions to assure the convergence of the approximate controls (see [7], [3], [4]).

An approximation scheme for equation (1.1) consists of discretization of the spatial variable and an appropriate approximation for the resulting differential-delay equation [2], [5]. The later is more crucial since it is well known that the energy dissipation in (1.1) exclusively depends on the property of the kernel $g(s)$. Fabiano and Ito [5] used a general spatial variable discretization together with the so-called averaging scheme of Banks and Burns [1] and a spline based scheme of Ito and Kappel [9]. The convergence of the schemes was proved by the Trotter-Kato Theorem [11]. In this paper, we will show that these approximation schemes preserve exponential stability uniformly. The paper is organized as follows. In section 2, we present the semigroup setting developed by Fabiano and Ito. Our contribution is to show that the resulting semigroup is exponentially stable under the conditions (g1)-(g3) imposed on the kernel $g(s)$. In section 3, we present our main result on the preservation of the uniform exponential stability of the approximate semigroups. A similar result for the nonuniform mesh averaging scheme is given in section 4. The notations used in this paper are the same as in [5] for consistency.

2 Semigroup Setting

Assume that V and H are a pair of Hilbert spaces with $V \subset H$, a continuous dense injection. Let V^* denote the dual space of V. We identify H with its dual so that $V \subset H = H^* \subset V^*$. Consider a symmetric sesquilinear form σ on V such that

$$\begin{aligned} |\sigma(u,v)| &\leq C\|u\|_V\|v\|_V \quad \text{for } u,v \in V && (2.1)\\ \sigma(u,u) &\geq \omega\|u\|_V^2 && (2.2) \end{aligned}$$

where $\omega > 0$. Let $A \in \mathcal{L}(V, V^*)$ be defined by

$$\sigma(u,v) = \langle Au, v\rangle_{V^*,V} \quad \text{for } u,v \in V. \tag{2.3}$$

Then, the restriction of A on H defines a positive definite and self-adjoint operator, which we call A again with

$$\mathcal{D}(A) = \{u \in V \ : \ Au \in H\}. \tag{2.4}$$

Furthermore, we have $\mathcal{D}(A^{\frac{1}{2}}) = V$, and

$$\sigma(u,v) = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle_H \quad \text{for } u,v \in V. \tag{2.5}$$

Thus, V can be equipped with a scalar inner product $\langle u,v\rangle_V = \sigma(u,v)$. Let $W = L_g^2(-r,0;V)$ be the Hilbert space of all V-valued, square integrable functions defined on the measure space $([-r,0], V, gds)$ equipped with norm

$$\|w\|_W^2 = \int_{-r}^0 g(s)\|w(s)\|_V^2 ds. \tag{2.6}$$

Define

$$v = \dot{u}, \quad w(t,s) = u(t) - u(t+s), \tag{2.7}$$

for $t > 0, s \in [-r,0]$. Then equation (1.1) can be written as an abstract evolution equation

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) + \mathrm{col}(0, f(t), 0) \tag{2.8}$$

on the Hilbert space $Z = V \times H \times W$ equipped with the norm

$$\|z\|_Z^2 = \alpha\|u\|_V^2 + \|v\|_H^2 + \|w\|_W^2. \tag{2.9}$$

Here $z = (u,v,w)^T$ and

$$\mathcal{A}z = \begin{pmatrix} v \\ -A\left(\alpha u + \int_{-r}^0 g(s)w(s)ds\right) \\ v + D_s w \end{pmatrix} \tag{2.10}$$

with

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in Z \;:\; \alpha u + \int_{-r}^{0} g(s)w(s)ds \in \mathcal{D}(A);\;\; v \in V;\;\; D_s w \in W;\;\; w(0) = 0 \right\}. \tag{2.11}$$

When the condition (g3) in section 1 is replaced by a weak condition: non-negativity of g', the following theorem is proved in [5].

Theorem 2.1 *The linear operator $\mathcal{A}$, defined by (2.10) and (2.11), generates a C_0-semigroup $S(t)$ on Z.*

It is well-known that the decay rate of the solutions to equation (1.1) depends on the decay rate of the history kernel $g(s)$. To have an exponentially stable $S(t)$, we must impose conditions on $g(s)$ stronger than non-negativity of its derivative. Condition (g2) serves for this purpose. It is easy to see that the weakly singular kernel of the form

$$g(s) = c\frac{e^{\gamma s}}{|s|^p}, \quad c, \gamma > 0,\; 0 \le p < 1, \tag{2.12}$$

satisfies conditions (g1)-(g3). Fabiano and Ito obtained some results on the exponential stability of $S(t)$ (see [6]) for the nonsingular exponential decay kernels. However, it is not clear whether their results include the cases of singular kernel. We will prove the following theorem

Theorem 2.2 *If the kernel $g(s)$ satisfies conditions (g1)-(g3), then the linear operator $\mathcal{A}$, defined by (2.10) and (2.11), generates an exponentially stable C_0-semigroup $S(t)$ on Z.*

Our proof relies on the following characteristic conditions [8] for an exponentially stable C_0-semigroup.

Theorem 2.3 *A C_0-semigroup $S(t) = e^{t\mathcal{A}}$ on the Hilbert space Z is exponentially stable if and only if*

$$\sup\{Re\lambda;\;\; \lambda \in \sigma(\mathcal{A})\} < 0 \tag{2.13}$$

and

$$\sup_{Re\lambda \ge 0} \|(\lambda - \mathcal{A})^{-1}\| < +\infty \tag{2.14}$$

hold. Here $\sigma(\mathcal{A})$ stands for the spectrum of $\mathcal{A}$.

We first prove a lemma.

Lemma 2.1

$$\langle \mathcal{A}z, z\rangle_Z = -\frac{1}{2}g(-r)\|w(-r)\|_V^2 - \frac{1}{2}\int_{-r}^0 g'(s)\|w(s)\|_V^2 ds \tag{2.15}$$

for any $z \in \mathcal{D}(\mathcal{A})$.

Proof: By a straight forward calculation, we have

$$\begin{aligned} \langle \mathcal{A}z, z\rangle_Z &= \alpha\langle u, v\rangle_V - \left\langle A(\alpha u + \int_{-r}^0 g(s)w(s)ds),\ v\right\rangle_H \\ &\quad + \langle v + D_s w, w\rangle_W \\ &= \int_{-r}^0 g(s)\langle D_s w, w\rangle_V ds. \end{aligned} \tag{2.16}$$

Since $D_s w \in W$, we know that $sg(s)\|D_s w\|_V^2$ is bounded near $s = 0$. Therefore, Integrating by parts of the last term in (2.16) yields (2.15). □

Proof of Theorem 2.2: We use the contradiction argument. Suppose the conclusion is not true. Then by Theorem 2.3 one of (2.13) and (2.14) must fail to hold. in both Consider first the case when (2.14) fails to hold. Then there exist a sequence of $\lambda_n \in C$ and a sequence of $z_n = (u_n, v_n, w_n) \in \mathcal{D}(\mathcal{A})$ with $Re\lambda_n \geq 0$ and $\|z_n\|_Z = 1$ such that (as $n \to \infty$)

$$\lim_{n\to\infty} \|(\lambda_n I - \mathcal{A})z_n\|_Z = 0, \tag{2.17}$$

which is equivalent to

$$\lambda_n u_n - v_n \to 0 \text{ in } V, \tag{2.18}$$

$$\lambda_n v_n + A(\alpha u_n + \int_{-r}^0 g(s)w_n(s)ds) \to 0 \text{ in } H, \tag{2.19}$$

$$\lambda_n w_n(s) - v_n - D_s w_n(s) \to 0 \text{ in } W. \tag{2.20}$$

Equation (2.17) also applies to the case when (2.13) is violated with λ_n, z_n being an eigenpair of $\mathcal{A}$. In both cases, it follows from (2.15) and (2.17) that

$$\begin{aligned} Re\langle(\lambda_n I - \mathcal{A})z_n, z_n\rangle_Z &= Re\lambda_n\|z_n\|_Z^2 - Re\langle \mathcal{A}z_n, z_n\rangle_Z \\ &= Re\lambda_n + \frac{1}{2}g(-r)\|w_n(-r)\|_V^2 \\ &\quad + \frac{1}{2}\int_{-r}^0 g'(s)\|w_n(s)\|_V^2 ds \to 0. \end{aligned} \tag{2.21}$$

Each term in (2.21) tends to zero since they are all nonnegative. Thus, we have

$$\mathrm{Re}\lambda_n \to 0, \quad g(-r)\|w_n(-r)\|_V^2 \to 0, \tag{2.22}$$

and, by condition (g3),

$$w_n(s) \to 0 \quad \text{in} \quad W, \tag{2.23}$$

which, combining with $\|z_n\|_Z = 1$, implies

$$\alpha\|u_n\|_V^2 + \|v_n\|_H^2 \to 1. \tag{2.24}$$

From (2.18) and the continuous injection of V into H it follows that $\lambda_n u_n - v_n$ also converges to zero in H. Thus

$$\lambda_n \langle u_n, v_n\rangle_H - \|v_n\|_H^2 \to 0. \tag{2.25}$$

Taking inner product of (2.19) with u_n in H, we have

$$\begin{aligned} &\lambda_n\langle v_n, u_n\rangle_H + \sigma(\alpha u_n + \int_{-r}^0 g(s)w_n(s)ds,\ u_n) \\ = {}&\lambda_n\langle v_n, u_n\rangle_H + \alpha\|u_n\|_V^2 + \int_{-r}^0 g(s)\langle w_n(s), u_n\rangle_V ds \to 0. \end{aligned} \tag{2.26}$$

By (2.23) and the following estimate

$$\begin{aligned} \left|\int_{-r}^0 g(s)\langle w_n(s), u_n\rangle_V ds\right| &\leq \|u_n\|_V \int_{-r}^0 g(s)\|w_n(s)\|_V ds \\ &\leq \|u_n\|_V \left(\int_{-r}^0 g(s)ds\right)^{\frac{1}{2}} \|w_n\|_W, \end{aligned} \tag{2.27}$$

the third term in (2.26) converges to zero. Adding the complex conjugate of (2.25) to (2.26) results in

$$2(Re\lambda_n)\langle v_n, u_n\rangle_H + \alpha\|u_n\|_V^2 - \|v_n\|_H^2 \to 0. \tag{2.28}$$

Therefore, it follows from (2.24), (2.28) and $Re\lambda_n \to 0$ that

$$\alpha\|u_n\|_V^2 \to \frac{1}{2}, \quad \|v_n\|_H^2 \to \frac{1}{2}. \tag{2.29}$$

Now we claim $|\lambda_n| \geq \delta_1 > 0$ for all n. Otherwise, from (2.18), $v_n \to 0$ in V (at least a subsequence), so does in H. A contradiction. Dividing (2.18) by λ_n and applying (2.29) yields

$$\left\|\frac{v_n}{\lambda_n}\right\|_V^2 \to \frac{1}{2\alpha}. \tag{2.30}$$

The rest of the proof is to show that (2.30) is a contradiction. We rewrite (2.20) as

$$w_n(s) - \frac{v_n}{\lambda_n} - \frac{1}{\lambda_n}D_s w_n(s) \to 0 \quad in \quad W. \tag{2.31}$$

Under the conditions imposed on $g(s)$, it is clear that $s^2 g$ and also $s^2 g'(s) \in L^1(-r,0)$. Then we can easily verify that $\frac{sv_n}{\lambda_n} \in W$. Now, taking inner product of (2.31) with $\frac{sv_n}{\lambda_n}$ in W yields

$$\langle w_n(s), \frac{sv_n}{\lambda_n}\rangle_W - \left\| \frac{v_n}{\lambda_n} \right\|_V^2 \int_{-r}^0 sg(s)ds - \frac{1}{\lambda_n}\int_{-r}^0 sg(s)\langle D_s w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds \to 0. \tag{2.32}$$

The first term in (2.32) converges to zero because of (2.23),(2.30). To estimate the third term in (2.32), we first integrate by parts to get

$$\begin{aligned}\int_{-r}^0 sg(s)\langle D_s w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds &= rg(-r)\langle w_n(-r), \frac{v_n}{\lambda_n}\rangle_V \\ &\quad - \int_{-r}^0 sg'(s)\langle w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds \\ &\quad - \int_{-r}^0 g(s)\langle w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds, \end{aligned} \tag{2.33}$$

then estimate each term on the right hand side of (2.33) as follows.

$$\left| rg(-r)\langle w_n(-r), \frac{v_n}{\lambda_n}\rangle_V \right| \le rg(-r)\|w_n(-r)\|_V \|\frac{v_n}{\lambda_n}\|_V \to 0; \tag{2.34}$$

$$\begin{aligned}&\left|\int_{-r}^0 sg'(s)\langle w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds\right| \le \int_{-r}^0 sg'(s)\|w_n(s)\|_V \|\frac{v_n}{\lambda_n}\|_V ds \\ &\le \left(\int_{-r}^0 s^2 g'(s)ds\right)^{\frac{1}{2}} \left(\int_{-r}^0 g'(s)\|w_n(s)\|_V^2 ds\right)^{\frac{1}{2}} \|\frac{v_n}{\lambda_n}\|_V \to 0; \end{aligned} \tag{2.35}$$

$$\begin{aligned}\left|\int_{-r}^0 g(s)\langle w_n(s), \frac{v_n}{\lambda_n}\rangle_V ds\right| &\le \int_{-r}^0 g(s)\|w_n(s)\|_V \|\frac{v_n}{\lambda_n}\|_V ds \\ &\le \|w_n\|_W \left(\int_{-r}^0 g(s)ds\right)^{\frac{1}{2}} \|\frac{v_n}{\lambda_n}\|_V \to 0. \end{aligned} \tag{2.36}$$

Thus the third term of (2.32) also converges to zero. Now the second term of (2.32) must converge to zero. This contradicts (2.30). □

3 Uniformly Exponentially Stable Approximations

In this section, we first present the approximation schemes of equation (1.1) introduced in [5]. Then we show that these schemes preserve the exponential stability uniformly. Let V^N be any sequence of finite dimensional subspaces of V. Assume that for any $u \in V$, there exists a sequence $u^N \in V^N$ such that

$$\|u^N - u\|_V \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

Define a continuous linear operator $A^N \; : \; V^N \to V^N$ by

$$\langle A^N u, v\rangle_H = \sigma(u,v) \quad \text{for} \;\; u, v \in V^N. \tag{3.2}$$

Thus, the spaces V^N complete the discretization of the spatial variable in the sense that they provide a sequence of finite dimensional subspaces of V and H, and (3.2) gives an approximation of the operator A. This can often be realized by choosing a standard finite element scheme. The approximation schemes for the resulting delay equation considered here are the averaging scheme (see [1]) and a spline-based scheme (see [9]). They all involve the discretization of the history interval $[-r, 0]$ and the approximation of the differential operator D_s. Let $B_i^M,\ i = 1, \cdots, M$ be the linear spline elements corresponding to the mesh $s_i^M = -ih,\ i = 0, \cdots, M$, where $h = r/M$; i.e.,

$$B_i^M(s) = \begin{cases} \frac{1}{h}(s - s_{i+1}^M) & \text{for} \;\; s_{i+1}^M \le s \le s_i^M \\ \frac{1}{h}(s_{i-1}^M - s) & \text{for} \;\; s_i^M \le s \le s_{i-1}^M \\ 0 & \text{elsewhere.} \end{cases} \tag{3.3}$$

We also denote the characteristic function on $[s_i^M, s_{i-1}^M)$ by $E_i^M,\ i = 1, \cdots, M$. For each positive integer N, M, define the subspaces of W by

$$\widetilde{W}^{N,M} \;\; = \;\; \left\{ w \in W \;\mid\; w = \sum_{i=1}^{M} b_i^M B_i^M, \;\; b_i^M \in V^N \right\} \tag{3.4}$$

$$W^{N,M} \;\; = \;\; \left\{ w \in W \;\mid\; w = \sum_{i=1}^{M} a_i^M E_i^M, \;\; a_i^M \in V^N \right\} \tag{3.5}$$

In order to approximate the operator D_s, we consider the sesquilinear form on $\widetilde{W}^{N,M} \times W^{N,M}$

$$a^{N,M}(w^{N,M}, h^{N,M}) = \langle D_s w^{N,M}, h^{N,M}\rangle_W \tag{3.6}$$

for $w^{N,M} \in \widetilde{W}^{N,M}$ and $h^{N,M} \in W^{N,M}$. Since $a^{N,M}$ is continuous, there exists a linear operator $\widetilde{D}^{N,M} : \widetilde{W}^{N,M} \to W^{N,M}$ such that

$$a^{N,M}(w^{N,M}, h^{N,M}) = \langle \widetilde{D}^{N,M} w^{N,M}, h^{N,M}\rangle_W. \tag{3.7}$$

A straight forward calculation shows that $\widetilde{D}^{N,M}$ is given by

$$\widetilde{D}^{N,M} w^{N,M} = \frac{1}{h} \sum_{i=1}^{M} (b_{i-1}^M - b_i^M) E_i^M \tag{3.8}$$

for $w^{N,M} = \sum_{i=1}^{M} b_i^M B_i^M$, where $b_i^M \in V^N$, $b_0^M = 0$. Finally, we introduce the following isomorphisms from $\widetilde{W}^{N,M}$ to $W^{N,M}$:

$$\begin{aligned} i_1^{N.M} w^{N,M} &= \sum_{i=1}^{M} b_i^M E_i^M, & (3.9) \\ i_2^{N.M} w^{N,M} &= \sum_{i=1}^{M} \frac{b_{i-1}^M + b_i^M}{2} E_i^M. & (3.10) \end{aligned}$$

Now we are ready to obtain approximations of D_s by defining operators $D_k^{N,M} : W^{N,M} \to W^{N,M}$ by

$$D_k^{N,M} = \widetilde{D}^{N,M} (i_k^{N,M})^{-1}, \quad k = 1, 2. \tag{3.11}$$

Thus, the construction of the approximation scheme is completed.

Let $Z^{N,M} = V^N \times V^N \times W^{N,M}$ equipped with the norm induced from the norm in Z. We approximate equation (2.8) by

$$\frac{d}{dt} z^{N,M}(t) = \mathcal{A}_k^{N,M} z^{N,M}(t) + \mathrm{col}(0, P_H^N f(t), 0) \tag{3.12}$$

where $z^{N,M}(t) = (u^N(t), v^N(t), w^{N,M}(t))$, P_H^N is the orthogonal projection from H onto V^N, and

$$\mathcal{A}_k^{N,M} z^{N,M} = \begin{pmatrix} v^N \\ -A^N \left(\alpha u^N + \int_{-r}^0 g(s) w^{N,M}(s) ds \right) \\ v^N + D_k^{N,M} w^{N,M} \end{pmatrix}. \tag{3.13}$$

The following lemma was proved in [5].

Lemma 3.1 *For* $z^{N,M} = (u^N, v^N, w^{N,M}) \in Z^{N,M}$ *with* $w^{N,M} = \sum_{i=1}^{M} a_i^M E_i^M$,

$$\begin{aligned} \langle \mathcal{A}_1^{N,M} z^{N,M}, z^{N,M} \rangle_Z &= -\frac{1}{2} \left(\sum_{i=1}^{M-1} \frac{1}{h} (g_i^M - g_{i+1}^M) \|a_i^M\|_V^2 + \frac{1}{h} g_M^M \|a_M^M\|_V^2 \right. \\ &\quad \left. + \sum_{i=1}^{M} \frac{1}{h} g_i^M \|a_i^M - a_{i-1}^M\|_V^2 \right), \end{aligned} \tag{3.14}$$

and

$$\langle \mathcal{A}_2^{N,M} z^{N,M}, z^{N,M} \rangle_Z = -\frac{1}{2}\left(\sum_{i=1}^{M-1} \frac{1}{h}(g_i^M - g_{i+1}^M)\|b_i^M\|_V^2 + \frac{1}{h} g_M^M \|b_M^M\|_V^2\right) \tag{3.15}$$

where $a_i^M = (b_{i-1}^M + b_i^M)/2,\ g_i^M = \int_{s_i^M}^{s_{i-1}^M} g(s)ds,\ i = 1, \cdots, M,\ b_0^M = 0.$

Lemma 3.1 implies the dissipativeness of $\mathcal{A}_k^{N,M}$. Thus they generate C_0-semigroups, $S_k^{N,M}$, of contractions on $Z^{N,M}$. In order to show the uniformly exponential stability for the above approximation schemes, we will show that $S_k^{N,M}(t)$ are uniform exponentially stable, i.e., there exists constants $C, \beta > 0$, independent of N, M, such that

$$\|S_k^{N,M}(t)\|_{\mathcal{L}(Z,Z)} \le Ce^{-\beta t} \quad \text{for } \ t > 0. \tag{3.16}$$

It is natural to expect that the technique used in the proof of Theorem 2.2 should also apply to the present case. Thus, the following theorem (see [10]) parallel to theorem 2.3 is needed.

Theorem 3.1 *Let $T_n(t)$ be a sequence of semigroups of contraction on the Hilbert spaces H_n and $\mathcal{A}_n$ be the corresponding infinitesimal generators. Then $T_n(t)$ are uniform exponentially stable if and only if the following two conditions hold:*

$$\sup_{n\in N} \{Re\lambda;\ \ \lambda \in \sigma(\mathcal{A}_n)\} = \sigma_0 < 0; \tag{3.17}$$

$$\sup_{Re\lambda \ge 0, n\in N} \left\{\left\|(\lambda I - \mathcal{A}_n)^{-1}\right\|\right\} < \infty. \tag{3.18}$$

Now we are ready to present our main result.

Theorem 3.2 *If the kernel $g(s)$ satisfies conditions (g1)-(g3), then the semigroup $S_k^{N,M}(t)$ generated by the linear operator $\mathcal{A}_k^{N,M}$ defined in (3.13), is uniformly exponentially stable, for $k = 1, 2$.*

Proof: We prove this theorem for the cases of $k = 1, 2$, respectively.

Case 1 (Averaging scheme):

Suppose the conclusion is not true. Then by Theorem 3.1 one of (3.17) and (3.18) must fail to hold. If (3.18) fails to hold, there exist a sequence of $\lambda^{N,M} \in C$ and a sequence of of $z^{N,M} = (u^N, v^N, w^{N,M}) \in \mathcal{D}(\mathcal{A}_1^{N,M})$ with $Re\lambda^{N,M} \ge 0$ and $\|z^{N,M}\|_Z = 1$ such that

$$\lim_{N,M\to\infty} \|(\lambda^{N,M} I - \mathcal{A}_1^{N,M}) z^{N,M}\|_Z = 0, \tag{3.19}$$

which is equivalent to

$$\begin{aligned} \lambda^{N,M}u^N - v^N &\to 0 \text{ in } V, &(3.20)\\ \lambda^{N,M}v_n + A^N(\alpha u^N + \int_{-r}^0 g(s)w^{N,M}(s)ds) &\to 0 \text{ in } H, &(3.21)\\ \lambda^{N,M}w^{N,M}(s) - v^N - D_1^{N,M}w^{N,M}(s) &\to 0 \text{ in } W. &(3.22)\end{aligned}$$

Equation (3.19) also applies to the case when (3.17) is violated with $\lambda^{N,M}, z^{N,M}$ being an eigenpair of $\mathcal{A}_1^{N,M}$. In both cases, it follows from Lemma 3.1 and (3.19) that

$$\begin{aligned} &Re\langle(\lambda^{N,M}I - \mathcal{A}_1^{N,M})z^{N,M}, z^{N,M}\rangle_Z = \\ &\qquad Re\lambda^{N,M}\|z^{N,M}\|_Z^2 - Re\langle\mathcal{A}_1^{N,M}z^{N,M}, z^{N,M}\rangle_Z \\ &= \quad Re\lambda^{N,M} + \frac{1}{2}\left(\sum_{i=1}^{M-1}\frac{1}{h}(g_i^M - g_{i+1}^M)\|a_i^M\|_V^2 + \frac{1}{h}g_M^M\|a_M^M\|_V^2\right. \\ &\qquad \left. + \sum_{i=1}^{M}\frac{1}{h}g_i^M\|a_i^M - a_{i-1}^M\|_V^2\right) \\ &\to \quad 0. \qquad (3.23)\end{aligned}$$

Each term in last expression of (3.23) converges to zero since they are all nonnegative.

Next we show that (3.23) implies

$$\|w^{N,M}\|_W^2 = \sum_{i=1}^{M} g_i^M\|a_i^M\|_V^2 \to 0. \qquad (3.24)$$

Since

$$\begin{aligned} \frac{(g_i^M - g_{i+1}^M)}{h} &= \int_{s_i^M}^{s_{i-1}^M}\frac{g(s) - g(s-h)}{h}ds \\ &= \int_{s_i^M}^{s_{i-1}^M} g'(\xi)ds \qquad (s_i^M \le \xi \le s_{i-1}^M) \\ &\ge \int_{s_i^M}^{s_{i-1}^M}\delta g(\xi)ds \qquad (\text{by } (g3)) \\ &\ge \delta\int_{s_i^M}^{s_{i-1}^M} g(s-h)ds = \delta g_{i+1}^M, \qquad (3.25)\end{aligned}$$

and

$$\sum_{i=1}^{M} g_i^M \|a_i^M\|_V^2 = \sum_{i=1}^{M-1} (g_i^M - g_{i+1}^M)\|a_i^M\|_V^2 + \sum_{i=1}^{M-1} g_{i+1}^M \|a_i^M\|_V^2 + g_M^M \|a_M^M\|_V^2, \tag{3.26}$$

we can see that (3.24) is an immediate result of (3.23),(3.25) and (3.26). Therefore, it follows from $\|z^{N,M}\|_Z = 1$ that

$$\alpha\|u^N\|_V^2 + \|v^N\|_V^2 \to 1. \tag{3.27}$$

By (3.20) and the continuous injection of V into H, $\lambda^{N,M}u^N - v^N$ also converges to zero in H. Thus

$$\lambda^{N,M}\langle u^N, v^N\rangle_H - \|v^N\|_H^2 \to 0. \tag{3.28}$$

Taking the inner product of (3.21) with u^N in H, we have

$$\begin{aligned} &\lambda^{N,M}\langle v^N, u^N\rangle_H + \sigma(\alpha u^N + \int_{-r}^0 g(s) w^{N,M}(s) ds,\ u^N) \\ &= \lambda^{N,M}\langle v^N, u^N\rangle_H + \alpha\|u^N\|_V^2 + \int_{-r}^0 g(s)\langle w^{N,M}(s), u^N\rangle_V ds \to 0. \end{aligned} \tag{3.29}$$

By (3.24) and the following estimate

$$\begin{aligned} \left|\int_{-r}^0 g(s)\langle w^{N,M}(s), u^N\rangle_V ds\right| &\le \|u^N\|_V \int_{-r}^0 g(s)\|w^{N,M}(s)\|_V ds \\ &\le \|u^N\|_V \left(\int_{-r}^0 g(s) ds\right)^{\frac{1}{2}} \|w^{N,M}\|_W, \end{aligned} \tag{3.30}$$

the third term in (3.29) converges to zero. Adding the complex conjugate of (3.28) to (3.29) results in

$$2(Re\lambda^{N,M})\langle v^N, u^N\rangle_H + \alpha\|u^N\|_V^2 - \|v^N\|_H^2 \to 0. \tag{3.31}$$

Therefore, it follows from (3.27),(3.31) and $Re\lambda^{N,M} \to 0$ that

$$\alpha\|u^N\|_V^2 \to \frac{1}{2}, \quad \|v^N\|_H^2 \to \frac{1}{2}. \tag{3.32}$$

Now we claim $|\lambda^{N,M}| \ge \delta_1 > 0$ for all N, M. Otherwise, from (3.20), $v^N \to 0$ in V (at least a subsequence), so does in H. A contradiction. Dividing (3.20) by $\lambda^{N,M}$ and applying (3.32) yields

$$\left\|\frac{v^N}{\lambda^{N,M}}\right\|_V^2 \to \frac{1}{2\alpha}. \tag{3.33}$$

The rest of the proof is to show that (3.33) is a contradiction. We rewrite (3.22) as

$$w^{N,M}(s) - \frac{v^N}{\lambda^{N,M}} - \frac{1}{\lambda^{N,M}} D_1^{N,M} w^{N,M}(s) \to 0 \quad in \;\; W \tag{3.34}$$

and take inner product with $\frac{sv^N}{\lambda^{N,M}} \in W$ in W to obtain

$$\langle w^{N,M}(s), \frac{sv^N}{\lambda^{N,M}} \rangle_W - \left\| \frac{v^N}{\lambda^{N,M}} \right\|_V^2 \int_{-r}^0 sg(s)ds - \frac{1}{\lambda^{N,M}} \int_{-r}^0 sg(s) \langle D_1^{N,M} w^{N,M}(s), \frac{v^N}{\lambda^{N,M}} \rangle_V ds \to 0. \tag{3.35}$$

The first term in (3.35) converges to zero because of (3.24). We will show that the third term in (3.35) also converges to zero. Therefore the second term in (3.35) must converge to zero, which gives a contradiction. In doing so, we have

$$\begin{aligned} & \left| \int_{-r}^0 sg(s) \langle D_1^{N,M} w^{N,M}, \frac{v^N}{\lambda^{N,M}} \rangle_V ds \right| \\ = & \left| \sum_{i=1}^{M-1} \frac{(sg)_i^M - (sg)_{i+1}^M}{h} \langle a_i^M, \frac{v^N}{\lambda^{N,M}} \rangle_V - \frac{(sg)_M^M}{h} \langle a_M^M, \frac{v^N}{\lambda^{N,M}} \rangle_V \right| \\ \le & \left\| \frac{v^N}{\lambda^{N,M}} \right\|_V \left| \sum_{i=1}^{M-1} \frac{(sg)_i^M - (sg)_{i+1}^M}{h} \|a_i^M\|_V + \frac{r(g)_M^M}{h} \|a_M^M\|_V \right|, \end{aligned} \tag{3.36}$$

where $(sg)_i^M = \int_{s_i^M}^{s_{i-1}^M} sg(s)ds$ for $i = 1, \cdots, M$. A simple calculation leads to

$$\begin{aligned} & \sum_{i=1}^{M-1} \frac{(sg)_i^M - (sg)_{i+1}^M}{h} \|a_i^M\|_V \\ = & \sum_{i=1}^{M-1} \left(\int_{s_i^M}^{s_{i-1}^M} \frac{g(s) - g(s-h)}{h} sds + \int_{s_i^M}^{s_{i-1}^M} g(s-h)ds \right) \|a_i^M\|_V \\ = & \int_{-r+h}^0 s \frac{g(s) - g(s-h)}{h} \sum_{i=1}^{M-1} \|a_i^M\|_V E_i^M ds + \sum_{i=1}^{M-1} g_{i+1}^M \|a_i^M\|_V \\ \le & \left(\int_{-r+h}^0 s^2 \frac{g(s) - g(s-h)}{h} ds \right)^{\frac{1}{2}} \times \end{aligned}$$

$$\left(\int_{-r+h}^{0} \frac{g(s)-g(s-h)}{h} \sum_{i=1}^{M-1} \|a_i^M\|_V^2 E_i^M ds\right)^{\frac{1}{2}}$$
$$+\left(\sum_{i=1}^{M-1} g_{i+1}^M\right)^{\frac{1}{2}} \left(\sum_{i=1}^{M-1} g_{i+1}^M \|a_i^M\|_V^2\right)^{\frac{1}{2}}. \tag{3.37}$$

For the two terms in (3.37), their first factors are bounded since

$$s^2 g'(s), g(s) \in L(-r, 0).$$

Their second factors converge to zero because of (3.23) and (3.25). Combining (3.23), (3.36) and (3.37) yields that the third term in (3.35) converges to zero.

Case 2 (Spline-based scheme):

The proof is very similar to the one for Case 1. Therefore, we will only point out the differences and omit the redundant arguments.

1. Instead of (3.23), we have

$$Re\lambda^{N,M} + \frac{1}{2}\left(\sum_{i=1}^{M-1} \frac{1}{h}(g_i^M - g_{i+1}^M)\|b_i^M\|_V^2 + \frac{1}{h} g_M^M \|b_M^M\|_V^2\right) \to 0, \tag{3.38}$$

   where $a_i^M = (b_{i-1}^M + b_i^M)/2, i = 1, \cdots, M,\ b_0^M = 0$. This again leads to

$$\|w^{N,M}\|_W^2 = \sum_{i-1}^{M} g_i^M \|a_i^M\|_V^2 \to 0. \tag{3.39}$$

2. The estimates (3.36)-(3.37) are still valid when a_i^M is replaced by b_i^M since we have (3.38) and (3.39).

Thus, the proof of the theorem is completed. □

4 Nonuniform Mesh

In order to obtain a faster convergence for the approximation scheme, Fabiano and Ito proposed a nonuniform mesh averaging scheme (see [5]). In stead of choosing the uniform mesh for the history interval $[-r, 0]$, they chose s_i^M in the following way

$$\int_{s_i^M}^{s_{i-1}^M} g(s)ds = \frac{1}{M}\int_{-r}^{0} g(s)ds, \qquad i = 1, \cdots, M-1, \tag{4.1}$$

and $s_0^M = 0, s_M^M = -r$. Numerical experiments have been in favor of this scheme over the previous two schemes. Following the steps in the proof of Theorem 3.2, it is not difficult to show that the nonuniform mesh averaging scheme also preserves exponential stability uniformly. We will only point out several key steps. Let's still use $\mathcal{A}_1^{N,M}$ defined in (3.13) for the approximate operator with the nonuniform mesh (4.1).

1. Parallel to Lemma 3.1, we now have

$$\begin{aligned}\langle \mathcal{A}_1^{N,M} z^{N,M}, z^{N,M}\rangle_Z &= \langle D_1^{N,M} w^{N,M}, w^{N,M}\rangle_W \\ &= \int_{-r}^0 g(s)\langle \sum_{i=1}^M \frac{a_{i-1}^M - a_i^M}{\alpha_i^M} E_i^M, \sum_{i=1}^M a_i^M E_i^M\rangle_V ds \\ &= -\frac{1}{2}\left(\sum_{i=1}^{M-1}\left(\frac{g_i^M}{\alpha_i^M} - \frac{g_{i+1}^M}{\alpha_{i+1}^M}\right)\|a_i^M\|_V^2 + \frac{g_M^M}{\alpha_M^M}\|a_M^M\|_V^2\right. \\ &\quad \left. + \sum_{i=1}^M \frac{g_i^M}{\alpha_i^M}\|a_i^M - a_{i-1}^M\|_V^2\right), \end{aligned} \tag{4.2}$$

where $\alpha_i^M = s_{i-1}^M - s_i^M$. The dissipativeness of $\mathcal{A}_1^{N,M}$ is an immediate result of $\alpha_{i+1}^M > \alpha_i^M$.

2. By a linear translation, we have

$$\begin{aligned}&\frac{g_i^M}{\alpha_i^M} - \frac{g_{i+1}^M}{\alpha_{i+1}^M} \\ &= \frac{1}{\alpha_i^M}\int_{s_i^M}^{s_{i-1}^M}\left(g(s) - g(\frac{\alpha_{i+1}^M}{\alpha_i^M}s + \frac{s_{i-1}^M s_{i+1}^M - (s_i^M)^2}{\alpha_i^M})\right) ds \\ &= \frac{1}{\alpha_i^M}\int_{s_i^M}^{s_{i-1}^M} g'(\xi)\left(s - \frac{\alpha_{i+1}^M}{\alpha_i^M}s - \frac{s_{i-1}^M s_{i+1}^M - (s_i^M)^2}{\alpha_i^M}\right) ds \\ &\geq \delta\int_{s_i^M}^{s_{i-1}^M} g(\xi)\left(s - \frac{\alpha_{i+1}^M}{\alpha_i^M}s - \frac{s_{i-1}^M s_{i+1}^M - (s_i^M)^2}{\alpha_i^M}\right) ds \\ &\geq \delta\int_{s_i^M}^{s_{i-1}^M} g(\frac{\alpha_{i+1}^M}{\alpha_i^M}s + \frac{s_{i-1}^M s_{i+1}^M - (s_i^M)^2}{\alpha_i^M})\frac{\alpha_{i+1}^M}{\alpha_i^M} ds \\ &= \delta\int_{s_{i+1}^M}^{s_i^M} g(s)ds = \delta g_i^M. \end{aligned} \tag{4.3}$$

Here we have used conditions (g1)-(g2). Thus, when (4.2) converges to zero, we again obtain

$$\|w^{N,M}\|_W \to 0. \tag{4.4}$$

3. To show the term in (3.35) converges zero, we combine the linear translation used in (4.2) and the estimate technique used in (3.36)-(3.37). The calculation is tedious but elementary and we omit the details.

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Communicated by John Burns