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On an Inverse Dynamic Problem for Goursat–Darboux System*

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Abstract

The problem of dynamical modelling of unknown distributed and boundary disturbances in Goursat–Darboux system is considered. The finite-step dynamical regularizing algorithms for solution of this problem are constructed. These algorithms work in real time on feedback scheme. The estimations of convergence degree in L_2, L_∞, C- spaces are obtained.

Key words: inverse problems, ill-posed problems, feedback control

AMS Subject Classifications: 35K99 93B40

1 Introduction

Wide class of inverse problems consists in determination (estimation) of unknown parameters of dynamical systems by not precise measurement of systems' state. An approach to solving such type problems on the basis of finite-step dynamical algorithms was proposed by Yu. S. Osipov and A. V. Kryazhimskii [1]. These algorithms use the input information at the finite number of points of a time interval and process this information between points. Outputs of these algorithms are approximate value of unknown parameters. The algorithms work in real time mode, i.e., they reconstruct the parameters simultaneously with the dynamic of the process. Note, that these algorithms possess property of physical realizability (Volterra property). The algorithms also possess the regularizing property: the smaller error of measurement and the smaller distance between time

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points are, the more precise is the reconstruction (in the appropriate sense).

The method [1] is based on the ideas of the theory of positional control [19] and the theory of ill-posed problems [20, 21]. According with this approach we construct an auxiliary controlled model which functions simultaneously with the original system. The model is controlled positionally (by feedback scheme): at every time moment the control is formed on the basis of informations on the model state and approximate measurements the original system state realized up to this time moment.

We construct the control low for this model on the basis of ideas of extremal shift in such a way that the control realization approximates (reconstructs) the unknown parameter of the original system in the appropriate sense. Hence we reduce the inverse problem to a direct problem for an auxiliary controlled model.

Dynamical algorithms of reconstruction of distributed and boundary controls for some classes linear and non-linear parabolic systems and variation inequalities were constructed in [2, 3, 4, 5]. In [2, 3, 7] problems of reconstruction of unknown coefficients of elliptical operators in parabolic, hyperbolic and elliptical systems were considered. The problem of dynamical reconstruction of unknown streams of disturbances was solved in [5]. Similar problems for hyperbolic systems were discussed in [6, 8, 9]. In [2, 4, 5, 6, 7, 9] some accuracy estimations of reconstruction problem for system with distributed and boundary disturbances are obtained. The analogous non-dynamical constructions for inverse problems based on theory of control and theory of parameter estimation of dynamical systems are worked out in [10]. The analogous problems was considered in [14, 15, 16, 17].

In the present paper dynamical reconstructing algorithms for Goursat–Darboux system are constructed. The Gursat–Darboux models describe processes in chemical reactors, sorbing (desorbing) processes and etc. [11, 12]. Some inverse problems in the a posteriori non-dynamical state are considered in [11]. The present paper is connected with [14, 15, 16, 17] and continue [8, 9]. In paper [8] the inverse problem for reconstruction of unknown distributed and boundary parameters of Goursat-Darboux system was solved. This problem was analogous to problem from present paper but one was solved with another conditions in the parameters of the system. In particular, in [8] the inverse problem with fuzzy coefficients of the system was considered. In the paper [9] the finite-dimensional approximation for the problem of [8] was constructed. Conditions of convergence the finite-dimensional algorithm's realization were worked out. The present paper continue [8, 9]. The constructive accuracy estimates of the algorithms and convergence of approximations in stronger metrices are obtained.

2 Statement of the Problem

Let us consider the boundary Goursat-Darboux problem:

$$\begin{cases} y_{tx} = f_1(t,x)y_t + f_2(t,x)y_x + f_3(t,x,y)u(t,x), \\ y(t_0,x) = \varphi(x), \quad x \in S, \quad (t,x) \in \Pi, \\ y(t,0) = \psi_1(t) + \int\limits_{t_0}^{t} \psi_2(\tau)v(\tau)d\tau, \quad t \in T, \\ \varphi(0) = \psi_1(t_0), \end{cases} \tag{2.1}$$

where $S = [0,l]$ is the variation segment of space variable x, $T = [t_0,\theta]$ is the time interval, $\Pi = [t_0,\theta] \times [0,l]$. Assume the following restrictions are satisfied: functions $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot)$ are continuous on Π and satisfy the Lipschitz condition by t (i.e. there exist constants $Lip_i \geq 0$ such that $\mid f_i(t_1,x) - f_i(t_2,x) \mid \leq Lip_i \mid t_1 - t_2 \mid$ for all $x \in S$, $t_1,t_2 \in T$); the function f_3 is continuous by all variables $(t,x,y) \in T \times S \times R$ and satisfies the Lipschitz condition by (t,y) (i.e. there exists a constant $Lip_3 \geq 0$ such that $\mid f_3(t_1,x,y_1) - f_3(t_2,x,y_2) \mid \leq Lip_3(\mid t_1 - t_2 \mid + \mid y_1 - y_2 \mid)$ for all $x \in S$, $t_1,t_2 \in T$, $y_1,y_2 \in R$); functions $\varphi(\cdot)$, $\psi_1(\cdot)$, $\psi_2(\cdot)$ belong to spaces $W^1_\infty(S)$, $W^2_\infty(T)$, $W^1_\infty(T)$ respectively; parameters of the system u and v (disturbances and controls) satisfy restrictions $u \in U$, $v \in V$. Here U is the set of all measurable (by Lebesgue) mappings $\Pi \to R$ taking values in convex closed bounded set $P_u \subset L_2(S)$ for almost all $t \in T$; V is the set of all measurable (by Lebesgue) mappings $T \to R$ taking values in convex closed bounded set $P_v \subset R$ for almost all $t \in T$. Sets $U \subset L_\infty(T;L_2(S))$ and $V \subset L_\infty(T;R)$ are convex and weak compact in spaces $L_2(T;L_2(S))$ and $L_2(T;R)$ respectively. Notation y_t, y_x, y_{yx} are used for corresponding generalized partial derivatives of the function $y = y(\cdot,\cdot)$ defined on $T \times S$. One can find definitions and properties of functional spaces, for example, in [13].

Under the above assumptions there exists a unique function $y = y(\cdot,\cdot) \in W^1_2(\Pi) \cap C(\Pi) \cap C(T;W^1_2(S))$ which satisfies the equation from (2.1) for almost all $(t,x) \in \Pi$ and boundary conditions from (2.1) for all $t \in T$, $x \in S$. The existence and uniqueness of this solution is proved in [18]. Systems ot the Goursat-Darboux type describe the processes of substances interaction in linear chemical reactor [12]. The function $y = y(t,x)$, $t \in T$, $x \in S$, describes the concentration one of the interaction substances. The coefficients of the equation (2.1) specify the chemical reactor and a chemical process. The parameters u and v characterize catalizators of the chemical process or reaction's speed. The simplest restriction on the parameters u and v can have the forms:

$$|u(t,x)| \leq Const, \quad |v(t)| \leq Const, \quad t \in T, \quad x \in S.$$

If we fixed a time moment $t \in T$ then the solution $y(t,\cdot)$, as the function of variable x, belongs to $W^1_2(S)$. We will call $y(t,\cdot)$ a state of the system at the time moment t, and the space $W_2^1(S)$ will be called a phase space of the system. In order to emphasize dependence y on u and v we will also write $y(\cdot;u,v)$. The solution $y = y(\cdot;u,v) : T \to W_2^1(S)$ of the boundary problem (2.1) is the motion of the system which corresponds to parameters $u \in U$ and $v \in V$. The set $Y = \{y = y(\cdot;u,v) : u \in U, v \in V\}$ is a compactum in the space $C(\Pi)$ and a weak compactum in $W_2^1(\Pi)$.

Let us describe the informative part of the problem. Let $y^* \in Y$ be the real (observable) motion of the system on the time interval $T = [t_0, \vartheta]$, $W_* = W(y^*)$ be the set of all pairs of parameters $(u,v) \in U \times V$ generating the motion y^* (this set is not empty and can content several elements). The measurement of the system state $y^*(t) = y^*(t,\cdot)$ is available at each current moment t and the result of the measurement $\xi(t) \in W_2^1(S)$ satisfies the estimation

$$\|\xi(t,\cdot) - y^*(t,\cdot)\|_{W_2^1(S)} \leq h, \quad t \in T. \tag{2.2}$$

Using results of measurement, we should restore (in real time) one of pairs $(u,v) \in W_*$. If W_* is one-element, then the true pair of parameters generating the motion y^* will be restored approximately.

We will suppose the following: the equation of the system is known; the initial state $y(t_0,\cdot)$ is known with error h in the metric of the space $W_2^1(S)$; disturbances u and v are satisfy the condition $u \in U$, $v \in V$ and sets U and V are given.

Let Ξ be the set of all mappings $\xi : T \to W_2^1(S)$, $\Xi_h(y^*)$ be the set of $\xi \in \Xi$ satisfying condition (2.2). We will say that an operator $D : \Xi \to U \times V$ possesses Volterra property if: $(D\xi_1)(t) = (D\xi_2)(t), \quad t_0 \leq t \leq \tau$, when $\xi_1(t) = \xi_2(t), \quad t_0 \leq t \leq \tau, \quad t_0 < \tau \leq \theta$. Here (u_*, v_*) is the element of the set $W(y^*)$ which has the minimal norm in the space $L_2 = L_2(T; L_2(S)) \times L_2(T; R)$.

Let us formalize the statement of the problem. Using a priori information about system (2.1), it is necessary to construct the family $(D_h)_{h>0}$ of Volterra operators $D_h : \Xi \to U \times V$ which possess the property

$$\sup\{\rho(D_h\xi, (u_*, v_*)) : \xi \in \Xi_h(y^*)\} \to 0 \quad \text{as} \quad h \to 0,$$

for arbitrary motion $y^* \in Y$. Here ρ is a metric of one of functional spaces $L_2(T; L_2(S)) \times L_2(T; R)$, $L_\infty(T_*, L_2(S)) \times L_\infty(T_*; R)$, $C(T_*; L_2(S)) \times C(T_*; R)$, $T_* = [t_*, \vartheta]$, $t_0 < t_* < \vartheta$.

3 Solution of the Inverse Problem

We will construct the solution of the inverse problem as finite-step dynamical algorithms FSDA [1]. Let us define formally the family of one parameter

FSDA $(D_h)_{h>0}$ by the following conditions

$$D_h = ((\tau_i^h)_{i=\overline{0,m}},\ (r_i^h)_{i=\overline{0,m-1}},\ (\rho_i^h)_{i=\overline{0,m-1}}), \tag{3.3}$$

where $m = m(h) \in N = \{1, 2, ...\}$, $h > 0$,
$({\tau_i}^h)_{i=\overline{0,m}}$ is the partition of segment T, $(t_0 = \tau_0^h < \tau_1^h < ... < \tau_m^h = \vartheta)$,

$$r_i^h : W_2^1(S) \times W_2^1(S) \to U[{\tau_i}^h, \tau_{i+1}^h) \times V[{\tau_i}^h, \tau_{i+1}^h), \quad i = \overline{0, m-1},$$

$$\rho_i^h : W_2^1(S) \times W_2^1(S) \times W_2^1(S) \to W_2^1(S) \times W_2^1(S), \quad i = \overline{0, m-1}.$$

Here $\Delta(h) = \max\{|\ \tau_{i+1}^h - \tau_i^h\ | : i = \overline{0, m-1}\}$, $U[t_1, t_2)$ is the set of all measurable mappings from $[t_1, t_2)$ into P_u, $V[t_1, t_2)$ is the set of all measurable mappings from $[t_1, t_2)$ into P_v. Let $\Delta = \Delta(h) \le C \min\{\tau_{i+1}^h - \tau_i^h : i = \overline{0, m-1}\}$ for some $C > 0$. For the sake of simplicity the index h in the notation will be omitted.

For FSDA (3.3) and the function $\xi : T \to W_2^1(S)$ we will call (D_h, ξ) - sequence the family of the elements $(u_i, v_i, z_i)_{i=\overline{0,m-1}}$ such that

$$(u_i, v_i) = r_i(\xi(\tau_i), z_i), \quad (z_{i+1}, w_{i+1}) = \rho_i(\xi(\tau_i), z_i, w_i),\ i = \overline{0, m-1};$$

$z_0 = \xi(t_0)$, w_0 is the solution on S of Cauchy problem for the ordinary differential equation

$$\frac{dw_0(x)}{dx} = f_1(t_0, x)w_0(x) + f_2(t_0, x)\xi_x(t_0, x) +$$
$$+ f_3(t_0, x, \xi(t_0, x))\tilde{u}_0(x), \quad x \in S,$$
$$w_0(0) = \psi_{1t}(t_0) + \psi_2(t_0)\tilde{v}_0,$$

where

$$\tilde{u}_0(x) = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} u_0(\tau, x)d\tau, \quad \tilde{v}_0 = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} v_0(\tau)d\tau$$

(obviously, the solution of this problem can be founded in the explicit form).

Pair of functions $(u, v) \in U \times V$ of the form $u(t) = u_i$, $v(t) = v_i$, $t \in [\tau_i, \tau_{i+1})$, $i = \overline{0, m-1}$ is called (D_h, ξ) - realization. This is the FSDA output, $D_h\xi = (u, v)$.

Let us describe, informally, how the FSDA (3.3) works in time. Before the initial moment t_0 the partition $(\tau_i)_{i=\overline{0,m}}$ is defined and fixed in accordance with the error value h. Every point τ_i will be the initial point of the next step of calculation. At moment $t = \tau_i$, $i = \overline{0, m-1}$, the input information $\xi(\tau_i)$ is supplied. Using this information and values z_i, w_i (of some auxiliary variables), at the moment $t = \tau_{i+1}$ new values of the auxiliary variables z_{i+1}, w_{i+1} by low ρ_i and the element $(u_i, v_i) = r_i(\xi(\tau_i), z_i) \in U[\tau_i, \tau_{i+1}) \times V[\tau_i, \tau_{i+1})$ by low r_i are determined. At the final time moment ϑ the (D_h, ξ) - realization (u, v), $D_h\xi = (u, v)$ is constructed. It is clear, that this operator D_h possesses Volterra property.

Let us determine now the concrete family FSDA (3.3).

The mapping $r_i(\lambda, z) = (u_i, v_i)$, $i = \overline{0, m-1}$, is constructed by the following rules: $u_i(t) = \tilde{u}_i,\ \tau_i \le t < \tau_{i+1}$, where

$$\tilde{u}_i = argmin\{2 < z_x(\cdot) - \lambda_x(\cdot), f_3(\tau_i, \cdot, \lambda(\cdot))u(\cdot) >_{L_2(S)} + \qquad (3.4)$$

$$+\alpha(h)\|u(\cdot)\|^2_{L_2(S)} : u(\cdot) \in P_u\};$$

$v_i(t) = \tilde{v}_i,\ \tau_i \le t < \tau_{i+1}$, where

$$\tilde{v}_i = argmin\{2 < z(0) - \lambda(0), \psi_2(\tau_i)v >_R + \qquad (3.5)$$

$$+\alpha(h) \mid v \mid^2 : v(\cdot) \in P_v\}.$$

Here $\alpha(\cdot)$ is a function $\alpha(\cdot) : [0, \infty) \to (0, \infty)$ (a parameter of the regularization).

Let us determine (ρ_i), $i = \overline{0, m-1}$, as: $\rho_i(\lambda, z, w) = (\tilde{z}(\tau_{i+1}, \cdot), \lambda)$, where

$$\tilde{z}(t, x) = z(x) + (t - \tau_i)\psi_2(\tau_i)\tilde{v}_i + \psi_1(t) - \psi_1(\tau_i) + \qquad (3.6)$$

$$+(t - \tau_i)\int_0^x (f_1(\tau_i, \eta)\Lambda_i(\eta) + f_2(\tau_i, \eta)\lambda_x(\eta) +$$

$$+f_3(\tau_i, \eta, \lambda(\eta))\tilde{u}_i(\eta))d\eta,\ t \in [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1},\ x \in [0, l],$$

$$z(t_0, x) = \xi(t_0, x).$$

Here

$$\Lambda_0[\lambda, w](\eta) = w_0(\eta), \quad \Lambda_i[\lambda, w](\eta) = \frac{w(\eta) - \lambda(\eta)}{\tau_{i-1} - \tau_i}, \quad i = \overline{1, m}.$$

Condition 1 *Functions u_*, v_* have generalized derivatives $u_{*t} \in L_\infty(\Pi)$, $v_{*t} \in L_\infty(T)$ and there exists the generalized derivative $y^*_{tt} \in L_\infty(\Pi)$.*

Condition 2 *There exist constants k_1 and k_2 such that for all $(t, x) \in \Pi$ and $y \in R$*

$$0 < k_1 \le \mid f_3(t, x, y) \mid \le k_2,$$

there exist constants k_3 and k_4 such that for every $t \in T$

$$0 < k_3 \le \mid \psi_2(t) \mid \le k_4.$$

Let us show that the family of FSDA (3.3)-(3.6) solves the inverse problem under the Conditions 1,2.

Lemma 1 *Let conditions 1 and 2 hold. Then for almost all* $t \in T$

$$\|z^*_{tx}(t,\cdot) - y^*_{tx}(t,\cdot)\|_{L_2(S)} \le a_1\alpha(h) + a_2 exp(\frac{k_1^2(t_0-t)}{\alpha(h)})(1+\alpha(h)),$$

$$\|z^*_{x}(t,\cdot) - y^*_{x}(t,\cdot)\|_{L_2(S)} \le \alpha(h)(M_1 + M_2 exp(\frac{k_1^2(t_0-t)}{\alpha(h)})),$$

$$| z^*_t(t,0) - y^*_t(t,0) | \le a_3\alpha(h) + a_4 exp(\frac{k_3^2(t_0-t)}{\alpha(h)})(1+\alpha(h)),$$

$$| z^*(t,0) - y^*(t,0) | \le \alpha(h)(M_3 + M_4 exp(\frac{k_3(t_0-t)}{\alpha(h)})),$$

where z^* *is the solution of the boundary problem*

$$\begin{cases} z^*_{tx} = f_1(t,x)y^*_t + f_2(t,x)y^*_x + f_3(t,x,y^*)\overline{u}_*(t,x), \\ z^*(t_0,x) = y^*(t_0,x), \quad x \in S, \quad (t,x) \in \Pi, \\ z^*(t,0) = \psi_1(t) + \int\limits_{t_0}^{t} \psi_2(\tau)\overline{v}_*(\tau)d\tau, \quad t \in T, \\ z^*(t_0,0) = y^*(t_0,0), \end{cases} \tag{3.7}$$

$$\overline{u}_*(t,x) = -\frac{f_3(t,x,y^*(t,x))}{\alpha(h)}(z^*_x(t,x) - y^*_x(t,x)),$$

$$\overline{v}_*(t) = -\frac{\psi_2(t)}{\alpha(h)}(z^*(t,0) - y^*(t,0)).$$

Constants a_i, M_i, $i = \overline{1,4}$, *depend only on a priori known parameters of the boundary problem (2.1).*

Proof: Let us introduce the following notation:

$$F(t,x) = f_1(t,x)y^*_t(t,x) + f_2(t,x)y^*_x(t,x),$$

$$B(t,x) = f_3(t,x,y^*(t,x)), \quad A(t,x) = B(t,x)B(t,x),$$

$$Z(\cdot,\cdot) = z^*_x(\cdot,\cdot) - y^*_x(\cdot,\cdot),$$

$$k_1 \le | B(t,x) | \le k_2, \quad k_2^{-1} \le | B(t,x)^{-1} | \le k_1^{-1},$$

$$\mathrm{ess}\sup_{(t,x)\in\Pi} | [B(t,x)^{-1}]_t | \le b,$$

$$\mathrm{ess}\sup_{t\in T} \|u_*(t,\cdot)\|_{L_2(S)} \le \gamma_u, \quad \mathrm{ess}\sup_{t\in T} \|u_{*t}(t,\cdot)\|_{L_\infty(S)} \le \overline{\gamma}_u.$$

It follows from properties of the function $f_3(\cdot)$ that there exists the generalized derivative $[B(\cdot)^{-1}]_t \in L_\infty(\Pi)$. From (2.1), (3.7) we have

$$z^*_{tx}(t,x) = -\frac{A(t,x)}{\alpha(h)}(z^*_x(t,x) - y^*_x(t,x)) + F(t,x),$$

$$Z_t(t,x) = -\frac{A(t,x)}{\alpha(h)}Z(t,x) - B(t,x)u_*(t,x), \quad Z(t_0,x) = 0. \tag{3.8}$$

By $X(\cdot,\cdot)$ we denote the solution of the following Cauchy problem

$$X_t(t,x) = -\frac{A(t,x)}{\alpha(h)}X(t,x), \quad X(t_0,x) = 1.$$

The function $X(\cdot,\cdot)$ is called the fundamental solution. Then the solution of (3.8) can be written as

$$Z(t,x) = -\int_{t_0}^{t} X(t,x)[X(\tau,x)]^{-1}B(\tau,x)u_*(\tau,x)d\tau. \tag{3.9}$$

Differentiating (3.9) with respect to (3.8) we have

$$Z_t(t,x) = -B(t,x)u_*(t,x)+ \tag{3.10}$$

$$+\frac{A(t,x)}{\alpha(h)}X(t,x)\int_{t_0}^{t}[X(\tau,x)]^{-1}B(\tau,x)u_*(\tau,x)d\tau.$$

Let us show that the integral in (3.10) has the following form

$$\int_{t_0}^{t}[X(\tau,x)]^{-1}B(\tau,x)u_*(\tau,x)d\tau = u_*(\tau,x)P(\tau,x)\Big|_{t_0}^{t} - \tag{3.11}$$

$$- -\int_{t_0}^{t} P(\tau,x)u'_{*t}(\tau,x)d\tau,$$

where

$$P(\tau,x) = \int_{t_0}^{\tau}[X(s,x)]^{-1}B(s,x)ds. \tag{3.12}$$

Note, that

$$\alpha(h)[X(t,x)^{-1}]_t[B(t,x)^{-1}] = [X(t,x)]^{-1}B(t,x), \tag{3.13}$$

$$P(\tau,x) = \alpha(h)([X(\tau,x)]^{-1}[B(\tau,x)]^{-1} - [B(t_0,x)]^{-1}) - \tag{3.14}$$

$$-\alpha(h)\int\limits_{t_0}^{\tau}[X(s,x)]^{-1}[B(s,x)^{-1}]_t ds.$$

Substituting (3.14) and (3.11) into (3.10) we obtain

$$Z_t(t,x) = -A(t,x)u_*(t,x)X(t,x)[B(t_0,x)]^{-1}-$$

$$--A(t,x)u_*(t,x)X(t,x)\int\limits_{t_0}^{t}[X(\tau,x)]^{-1}[B(\tau,x)^{-1}]_t d\tau-$$

$$--\frac{A(t,x)}{\alpha(h)}X(t,x)\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}[X(s,x)]^{-1}B(s,x)u_{*t}(\tau,x)dsd\tau.$$

The following estimate for almost all $t\in T$ is valid

$$\|Z_t(t,\cdot)\|_{L_2(S)} \le 3k_2^2k_1^{-1}\gamma_u exp(\frac{k_1^2(t_0-t)}{\alpha(h)})+$$

$$+3\alpha(h)k_2^2k_1^{-2}\gamma_u b \mid 1-exp(\frac{k_1^2(t_0-t)}{\alpha(h)}) \mid +$$

$$+3k_2^3\sqrt{l}\overline{\gamma}_u(k_1^{-2}mes(T)exp(\frac{k_1^2(t_0-t)}{\alpha(h)})+$$

$$+k_1^{-4}\alpha(h) \mid 1-exp(\frac{k_1^2(t_0-t)}{\alpha(h)}) \mid) \le$$

$$\le a_1\alpha(h)+a_2 exp(\frac{k_1^2(t_0-t)}{\alpha(h)})(1+\alpha(h))$$

$$\|Z(t,\cdot)\|_{L_2(S)} \le \alpha(h)k_1^{-2}k_2(\gamma_u+mes(T)\sqrt{l}\overline{\gamma}_u) \mid 1-exp(\frac{k_1^2(t_0-t)}{\alpha(h)}) \mid\le$$

$$\le \alpha(h)(M_1+M_2 exp(\frac{k_1^2(t_0-t)}{\alpha(h)})).$$

All constants depend only on a priori known parameters of the boundary problem (2.1). Third and fourth inequalities of the lemma can be obtained analogously.

□

Lemma 2 *Let conditions 1 and 2 hold and the function $\xi\in\Xi_h(y^*)$. Then for all $t\in T$ the following estimations are valid*

$$\|\overline{z}_x(t,\cdot)-z_x^*(t,\cdot)\|_{L_2(S)} \le L(\sqrt{\alpha(h)}\ \nu(\Delta,h)+h), \qquad (3.15)$$

$$| \overline{z}(t,0) - z^*(t,0) | \leq L(\Delta + h), \tag{3.16}$$

where

$$\nu(\Delta, h) = \frac{h}{\Delta} + \frac{h}{\alpha(h)} + \sigma_q(\Delta) + \sigma_p(\Delta) + \frac{\Delta}{\alpha(h)} + \Delta + h + \frac{\sigma_p(\Delta)}{\alpha(h)},$$

$$\sigma_q(\Delta) = \max_{i=\overline{1,m}} \max_{t\in[\tau_{i-1},\tau_i)} \|\Lambda_i[y^*(\tau_i,\cdot), y^*(\tau_{i-1},\cdot)](\cdot) - y_t^*(t,\cdot)\|_{L_2(S)},$$

$$\sigma_p(\Delta) = \max_{i=\overline{0,m-1}} \max_{t\in[\tau_i,\tau_{i+1})} \|y_x^*(\tau_i,\cdot) - y_x^*(t,\cdot)\|_{L_2(S)},$$

z^* is determined by (3.7), $\overline{z}$ is determined by equality

$$\overline{z}(t,x) = z(\tau_i,x) + (t-\tau_i)\psi_2(\tau_i)\overline{v}_i + \psi_1(t) - \psi_1(\tau_i) +$$

$$+(t-\tau_i)\int_0^x (f_1(\tau_i,\eta)\Lambda_i[\xi(\tau_i,\cdot),\xi(\tau_{i-1},\cdot)](\eta) + f_2(\tau_i,\eta)\xi_x(\tau_i,\eta) +$$

$$+f_3(\tau_i,\eta,\xi(\eta))\overline{u}_i(\eta))d\eta,$$

$$\overline{z}(t_0,x) = \xi(t_0,x), \quad x\in[0,l], \quad t\in[\tau_i,\tau_{i+1}), \quad i=\overline{0,m-1},$$

$$\overline{u}_i(t,x) = -\frac{f_3(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}(\overline{z}_x(\tau_i,x) - \xi_x(\tau_i,x)),$$

$$\overline{v}_i(t) = -\frac{\psi_2(\tau_i)}{\alpha(h)}(\overline{z}(\tau_i,0) - \xi(\tau_i,0)), \quad t\in[\tau_i,\tau_{i+1}), \quad i=\overline{0,m-1}.$$

The constant $L>0$ depends only on a priori known parameters of the boundary problem (2.1).

Proof: Let us introduce the following notation:

$$A(t,x) = f_3(t,x,y^*(t,x))f_3(t,x,y^*(t,x)), \quad Z = \overline{z}_x - z_x^*.$$

Taking into account the forms of functions $\overline{z}$, z^* and parameters ($\overline{u}_i$, $\overline{v}_i$, $\overline{u}_*$, $\overline{v}_*$) it follows that

$$\overline{z}_{tx}(t,x) = f_1(\tau_i,x)\Lambda_i[\xi(\tau_i,\cdot),\xi(\tau_{i-1},\cdot)](x) + f_2(\tau_i,x)\xi_x(\tau_i,x) -$$

$$- - \frac{f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}(\overline{z}_x(\tau_i,x) - \xi_x(\tau_i,x)), \; t\in[\tau_i,\tau_{i+1}), \; i=\overline{0,m-1},$$

$$\overline{z}_x(t_0,x) = \xi_x(t_0,x), \quad x\in S;$$

$$z_{tx}^*(t,x) = f_1(t,x)y_t^*(t,x) + f_2(t,x)y_x^*(t,x) -$$

$$- - \frac{A(t,x)}{\alpha(h)}(z_x^*(t,x) - y_x^*(t,x)), \quad t\in T,$$

$$z_x^*(t_0,x)=y_x^*(t_0,x),\quad x\in S.$$

Then for almost all $t\in[\tau_i,\tau_{i+1}),\quad i=\overline{0,m-1}$, we have

$$Z_t(t,x)=-\frac{A(t,x)}{\alpha(h)}Z(t,x)+\Phi(t,x),\quad Z(t_0,x)=\xi_x(t_0,x)-y_x^*(t_0,x),$$

where

$$\Phi(\cdot,\cdot)=S_1(\cdot,\cdot)+S_2(\cdot,\cdot)+S_3(\cdot,\cdot)+S_4(\cdot,\cdot),$$

$$S_1(t,x)=f_1(\tau_i,x)\Lambda_i[\xi(\tau_i,\cdot),\xi(\tau_{i-1},\cdot)](x)-f_1(t,x)y_t^*(t,x),$$

$$S_2(t,x)=f_2(\tau_i,x)\xi_x(\tau_i,x)-f_2(t,x)y_x^*(t,x),$$

$$S_3(t,x)=\frac{A(t,x)}{\alpha(h)}\overline{z}_x(t,x)-\frac{f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}z_x(\tau_i,x),$$

$$S_4(t,x)=\frac{f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}\xi_x(\tau_i,x)-\frac{A(t,x)}{\alpha(h)}y_x^*(t,x).$$

The functions f_i, $i=1,2,3$, satisfy the Lipschitz condition, the norms $\|y_t\|_{L_\infty(T;L_2(S))}$, $\|y_x\|_{L_\infty(T;L_2(S))}$ are bounded by a constant depending only a priori known parameters, hence from inequality (2.2) we have

$$S_1(t,x)=f_1(\tau_i,x)(\Lambda_i[\xi(\tau_i,\cdot),\xi(\tau_{i-1},\cdot)](x)-\Lambda_i[y^*(\tau_i,\cdot),y^*(\tau_{i-1},\cdot)](x)+$$

$$+\Lambda_i[y^*(\tau_i,\cdot),y^*(\tau_{i-1},\cdot)](x)-y^*(t,x))-(f_1(t,x)-f_1(\tau_i,x))y_t^*(t,x),$$

$$\|S_1(t,\cdot)\|_{L_2(S)}\le L_1(\frac{h}{\Delta}+\sigma_q(\Delta)+\Delta),\quad t\in T,$$

$$S_2(t,x)=f_2(\tau_i,x)(\xi_x(\tau_i,x)-y_x^*(\tau_i,x)+(y_x^*(\tau_i,x)-y_x^*(t,x))-$$

$$-(f_2(t,x)-f_2(\tau_i,x))y_x^*(t,x),$$

$$\|S_2(t,\cdot)\|_{L_2(S)}\le L_2(h+\sigma_p(\Delta)+\Delta),\quad t\in T;$$

$$S_3(t,x)=\frac{A(t,x)-f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}\overline{z}_x(t,x)+$$

$$+\frac{f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}(\overline{z}_x(t,x)-\overline{z}_x(\tau_i,x)),$$

$$\|S_3(t,\cdot)\|_{L_2(S)}\le L_3(\frac{\Delta+h}{\alpha(h)}+\frac{\Delta}{\alpha(h)}),\quad t\in T;$$

$$S_4(t,x)=-\frac{A(t,x)-f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}y_x^*(t,x)-$$

$$-\frac{f_3^2(\tau_i,x,\xi(\tau_i,x))}{\alpha(h)}(y_x^*(t,x)-\xi_x(\tau_i,x)),$$

$$\|S_4(t,\cdot)\|_{L_2(S)} \leq L_4(\frac{\Delta+h}{\alpha(h)} + \frac{\sigma_p(\Delta)}{\alpha(h)} + \frac{h}{\alpha(h)}), \quad t \in T;$$

$$\|\Phi(t,\cdot)\|_{L_2(S)} \leq L_*(\frac{h}{\Delta} + \frac{h}{\alpha(h)} + \sigma_q(\Delta) + \sigma_p(\Delta) + \frac{\Delta}{\alpha(h)} + \tag{3.17}$$

$$\Delta + h + \frac{\sigma_p(\Delta)}{\alpha(h)}) = L_*\nu(\Delta,h),$$

where $L_* = \max\{L_1, L_2, L_3, L_4\}$. Using Cauchy formula one can obtain

$$Z(t,x) = -\int_{t_0}^{t} X(t,x)[X(\tau,x)]^{-1}\Phi(\tau,x)d\tau +$$

$$+X(t,x)[X(t_0,x)]^{-1}(\xi_x(t_0,x) - y_x^*(t_0,x)).$$

Then

$$\|Z(t,\cdot)\|_{L_2(S)} \leq L(\sqrt{\alpha(h)}\ \nu(\Delta,h) + h), \quad t \in T. \tag{3.18}$$

One can prove analogously that

$$\overline{z}(t,0) = z(\tau_i,0) + \psi_1(t) - \psi_1(\tau_i) - (t-\tau_i)\frac{\psi_2^2(\tau_i)}{\alpha(h)}(\overline{z}(\tau_i,0) - \xi(\tau_i,0)),$$

$$z^*(t,0) = z^*(\tau_i,0) + \psi_1(t) - \psi(\tau_i) - \frac{1}{\alpha(h)}\int_{\tau_i}^{t}\psi_2^2(\tau)(z^*(\tau,0) - y^*(\tau,0))d\tau,$$

$$\overline{z}_t(t,0) - z_t^*(t,0) = --\frac{\psi_2^2(t)}{\alpha(h)}(\overline{z}(t,0) - z^*(t,0)) + \Psi(t),$$

$$\overline{z}(0,0) - z^*(0,0) = 0,$$

where

$$\Psi(t) = -\frac{\psi_2^2(\tau_i)}{\alpha(h)}\overline{z}(\tau_i,0) + \frac{\psi_2^2(t)}{\alpha(h)}\overline{z}(t,0) + \frac{\psi_2^2(\tau_i)}{\alpha(h)}\xi(\tau_i,0) - \frac{\psi_2^2(t)}{\alpha(h)}y^*(t,0).$$

Then

$$\mid \Psi(t) \mid \leq L_5(\frac{\Delta}{\alpha(h)} + \frac{h}{\alpha(h)}). \tag{3.19}$$

The previous inequality and properties of the solution of the problem (2.1) imply the inequality

$$\mid \overline{z}(t,0) - z^*(t,0) \mid \leq L(h+\Delta), \quad t \in T. \quad \Box \tag{3.20}$$

Lemma 3 *Let (u,v) be (D_h,ξ)-realization of FSDA (3.3)-(3.6). Then for almost all $t\in T$ the following inequality is valid*

$$\|u(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}+\mid v(t)-v_*(t)\mid\leq$$

$$\leq\|\overline{u}(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}+\mid\overline{v}(t)-v_*(t)\mid.$$

The proof of this lemma follows from the facts that (u,v) is the nearest element from $P_u\times P_v$ to the point $(\overline{u},\overline{v})$ in the metric of the space $L_2(S)\times R$ and the set $P_u\times P_v$ is convex.

Lemma 4 *Let conditions 1, 2 hold and $\xi\in\Xi_h(y^*)$. Then for almost all $t\in T$ the following inequality*

$$\|\overline{u}(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}+\mid\overline{v}(t)-v_*(t)\mid\leq\mu(h,t),$$

is valid, where

$$\mu(h,t)=\overline{L}[\Delta+h+\alpha(h)+\frac{\nu(\Delta,h)}{\sqrt{\alpha(h)}}+(1+\alpha(h))\times$$

$$\times(exp(\frac{k_1^2(t_0-t)}{\alpha(h)})+exp(\frac{k_3^2(t_0-t)}{\alpha(h)}))],$$

and $\overline{L}$ is some positive constant, depending on a priori known parameters of boundary problem (2.1).

Proof: For almost all $t\in[\tau_i,\tau_{i+1})$, $i=\overline{0,m-1}$, we have

$$\|\overline{u}(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}+\mid\overline{v}(t)-v_*(t)\mid\leq$$

$$\leq\|\overline{u}(t,\cdot)-\overline{u}_*(t,\cdot)\|_{L_2(S)}+\|\overline{u}_*(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}+$$

$$+\mid\overline{v}(t)-\overline{v}_*(t)\mid+\mid\overline{v}_*(t)-v_*(t)\mid\leq$$

$$\leq\|-\frac{f_3(\tau_i,\cdot,\xi(\tau_i,\cdot))}{\alpha(h)}(\overline{z}_x(\tau_i,\cdot)-\xi_x(\tau_i,\cdot))+$$

$$+\frac{B(t,\cdot)}{\alpha(h)}(z_x^*(t,\cdot)-y_x^*(t,\cdot))\|_{L_2(S)}+$$

$$+\mid-\frac{\psi_2(\tau_i)}{\alpha(h)}(\overline{z}(\tau_i,0)-\xi(\tau_i,0))+\frac{\psi_2(t)}{\alpha(h)}(z^*(t,0)-y^*(t,0))\mid+$$

$$+\|\overline{u}_*(t,\cdot)-u_*(t,\cdot)\|_{L_2(S)}++\mid\overline{v}_*(t)-v_*(t)\mid\leq$$

$$\leq\|-\frac{f_3(\tau_i,\cdot,\xi(\tau_i,\cdot))}{\alpha(h)}[(\overline{z}_x(\tau_i,\cdot)-z_x^*(t,\cdot))-(\xi_x(\tau_i,\cdot)-y_x^*(t,\cdot))]\|_{L_2(S)}+$$

$$+\| \frac{1}{\alpha(h)}(f_3(t,\cdot,y^*(t,\cdot)) - f_3(\tau_i,\cdot,\xi(\tau_i,\cdot)))(z_x^*(t,\cdot) - y_x^*(t,\cdot))\|_{L_2(S)} +$$

$$+ \mid -\frac{\psi_2(\tau_i)}{\alpha(h)}[(\overline{z}(\tau_i,0) - z^*(t,0)) - (\xi(\tau_i,0) - y^*(t,0))] \mid +$$

$$+ \mid \frac{1}{\alpha(h)}(\psi_2(t) - \psi_2(\tau_i))(z^*(t,0) - y^*(t,0)) \mid +$$

$$+\|[B(t,\cdot)]^{-1}(z_{tx}^*(t,\cdot) - y_{tx}^*(t,\cdot))\|_{L_2(S)} + \mid [\psi_2(t)]^{-1}(z_t^*(t,0) - y_t^*(t,0)) \mid$$

$$\leq \tilde{L}[\frac{h}{\alpha(h)} + \frac{\nu(\Delta,h)}{\sqrt{\alpha(h)}} + \frac{\sigma_p(\Delta)}{\alpha(h)} + (\Delta + h)(M_1 + M_2 exp(\frac{k_1^2(t_0 - t)}{\alpha(h)})) +$$

$$+(\frac{\Delta}{\alpha(h)} + \frac{h}{\alpha(h)} + \frac{\sigma_p(\Delta)}{\alpha(h)} + \Delta(M_3 + M_4 exp(\frac{k_3^2(t_0 - t)}{\alpha(h)})) + a_1\alpha(h) +$$

$$+a_2 exp(\frac{k_1^2(t_0 - t)}{\alpha(h)})(1 + \alpha(h)) + a_3\alpha(h) + a_4 exp(\frac{k_3^2(t_0 - t)}{\alpha(h)})(1 + \alpha(h))]$$

$$\leq \overline{L}[\Delta + h + \alpha(h) + \frac{\Delta}{\alpha(h)} + \frac{h}{\alpha(h)} + \frac{\sigma_p(\Delta)}{\alpha(h)} + \frac{\nu(\Delta,h)}{\sqrt{\alpha(h)}} +$$

$$+(1 + \alpha(h))(exp(\frac{k_1^2(t_0 - t)}{\alpha(h)}) + exp(\frac{k_3^2(t_0 - t)}{\alpha(h)}))]. \quad \square$$

Theorem 1 *Let conditions 1, 2 hold, $\xi \in \Xi_h(y^*)$ and parameters of FSDA (3.3)-(3.6) satisfy the conditions $\Delta(h) \to 0$, $\alpha(h) \to 0$, $\frac{\nu(\Delta(h),h)}{\sqrt{\alpha(h)}} \to 0$ as $h \to 0$. Then for almost all $t \in T$*

$$\|(D_h\xi)(t,\cdot) - (u_*(t,\cdot), v_*(t))\|_{L_2(S)\times R} \leq \mu(h,t), \ \mu(h,t) \to 0 \ as \ h \to 0,$$

$$\sup\{\rho(D_h\xi,(u_*,v_*)) : \xi \in \Xi_h(y^*)\} \leq \overline{\mu}(h,t_*),$$

$$\overline{\mu}(h,t_*) = \mu(h,t_*) + \overline{L}\sqrt{3}(1 + \alpha(h))\sqrt{\alpha(h)}(\frac{1}{k_1} + \frac{1}{k_3}) \to 0 \ as \ h \to 0,$$

where ρ is the metric of the space $L_2(T;L_2(S)) \times L_2(T;R)$ or the space $L_\infty([t_*,\vartheta];L_2(S)) \times L_\infty([t_*,\vartheta];R)$, t_* is an arbitrary fixed value $t_0 < t_* < \vartheta$.

The proof of this theorem follows from lemmas 1-4.

Now, let us prove the statement on reconstruction of disturbances in the metric of the space $L_2(\Pi) \times L_2(T)$ for another type of u_* and v_*.

Condition 3 *Functions $u_* : T \ni t \to u_*(t,\cdot) \in L_2(S)$ and $v_* : T \ni t \to v_*(t) \in R$ have the bounded variations on T.*

Theorem 2 *Let conditions 2,3 hold,* $\xi \in \Xi_h(y^*)$ *and parameters of FSDA (3.3)-(3.6) satisfy the conditions* $\Delta(h) \to 0$, $\alpha(h) \to 0$, $\dfrac{\nu(\Delta(h),h)}{\sqrt{\alpha(h)}} \to 0$ *as* $h \to 0$. *Then*

$$\sup\{\rho(D_h\xi,(u_*,v_*)) : \xi \in \Xi_h(y^*)\} \leq \mu_*(h), \quad \mu_*(h) \to 0 \quad as \quad h \to 0,$$

where ρ *is the metric of the space* $L_2(T;L_2(S)) \times L_2(T;R)$,

$$\mu_*(h) = \overline{L}\sqrt{3}[\Delta(h) + h + \alpha(h) + \frac{\nu(\Delta(h),h)}{\sqrt{\alpha(h)}} + (1+\alpha(h))\sqrt{\alpha(h)}(\frac{1}{k_1} + \frac{1}{k_3})].$$

One can prove this theorem by analogy with theorem 1.

Remark 1 *One can find the finite dimension approximation of the inverse problem in [9].*

Remark 2 *(On convergence in the space* C*).*
Let us define $(D_h,\xi) = (u,v)$ *- realization by the following conditions*

$$u(t,\cdot) = \tilde{u}_i(\cdot) + \frac{\tilde{u}_{i+1}(\cdot) - \tilde{u}_i(\cdot)}{\tau_{i+1} - \tau_i}(t - \tau_i), \quad t \in [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1},$$

$$v(t) = \tilde{v}_i + \frac{\tilde{v}_{i+1} - \tilde{v}_i}{\tau_{i+1} - \tau_i}(t - \tau_i), \quad t \in [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1}.$$

Then if the conditions of the theorem 1 hold we have

$$\sup\{\rho(D_h\xi,(u_*,v_*)) : \xi \in \Xi_h(y^*)\} \to 0 \quad as \quad h \to 0,$$

where ρ *is the metric of the space* $C([t_*,\vartheta];L_2(S)) \times C([t_*,\vartheta];R)$, t_* *is an arbitrary fixed value* $t_0 < t_* < \vartheta$. *The validity of this statement follows from the form of* (D_h,ξ)*-realization and can be proved analogously to theorem 1.*

Remark 3 *If the condition 2 is not satisfied (for example, functions* f_3 *and* ψ_2 *are equal to zero on some set of positive measure), then the observed motion of the system can be generated by several parameters. Following to the constructions of the present paper we can find a simple dynamical procedure for reconstruction (in Hausdorff metric) of the whole set of generating parameters* (u,v). *Let, for example, function* $f_3 = f_3(t,x)$ *be continuous in* Π*;* W_* *be the set of all pairs of parameters* $(u,v) \in U \times V$ *generating the observed motion* y_**;* (u_h,v_h) *be reconstructed by the described law parameters and let*

$$W_h[\xi] = \{(u,v) \in U \times V \ : \ f_3 \cdot u = f_3 \cdot u_h, \ \psi_2 \cdot v = \psi_2 \cdot v_h\}.$$

Then if $\Delta(h)$ and $\alpha(h)$ satisfy the conditions of theorem 1 or theorem 2 we have

$$\sup\{\rho_H(W_h[\xi], W_*) : \xi \in \Xi_h(y_*)\} \to 0 \quad as \quad h \to 0,$$

where ρ_H is Hausdorff metric on corresponding subsets of $U \times V$ generated by the metric of the space $L_2(T; L_2(S)) \times L_2(T; R)$.

Remark 4 *Let conditions 1, 2 be fulfilled and additional conditions*

$$f_1 \equiv 0, \quad f_2 \equiv 0, \quad f_3 \equiv 1, \quad \psi_2 \equiv 1$$

be hold. Then the family FSDA (3.3)-(3.6) with parameters $\alpha(h) = h^{1/2}$, $\Delta(h) = h$ has the asymptotical accuracy equal to $1/4$ in the metric of space L_2 and there exist positive constants C_1 and C_2 such that for all sufficiently small $h > 0$ the following inequalities are valid

$$C_1 \cdot h^{1/4} \leq \sup\{\rho_{L_2}(D_h\xi, (u_*, v_*)) : \xi \in \Xi_h(y^*)\} \leq C_2 \cdot h^{1/4}.$$

In this case, the problem under consideration is identical to the problem of numerical differentiation. It is known that the degree of the optimal algorithms of dynamical differentiation is equal to $1/3$. The degree of our algorithm is equal to $1/4$. However, our approach allows to construct the algorithms of numerical differentiation functioning in real time mode (synchro with the process) by the feedback principle. This approach is useful in some technical processes and the operating data processing [1].

Example 1. Let us consider the concrete variant of the reconstruction problem. The unknown parameters are distributed disturbances (controls). Let the system have the form

$$\begin{cases} y_{tx}(t,x) = y_x(t,x) + y(t,x) + \sin(x), & (t,x) \in [0,1] \times [0,1], \\ y(0,x) = 0, & x \in [0,1], \\ y(t,0) = 0, & t \in [0,1]. \end{cases}$$

The following restriction holds: $\max_{t \in T} \|u(t,\cdot)\|_{L_2(S)} \leq 1$. Let us choose a priori partition $(\tau_i)_{i=\overline{0,m}}$ such that $\Delta(h) =\mid \tau_{i+1} - \tau_i \mid = \frac{1}{m}$, $i = \overline{0, m-1}$. The point $t_* = 0.1$. In the table the results of the numerical experiment are carried out.

h	m	$\Delta(h)$	$\alpha(h)$	$\|u^h - u_*\|_{L_2}$	$\|u^h - u_*\|_{L_\infty}$
0	10	0.1	$\Delta^{1/2}$	0.3031	0.4835
0	50	0.02	$\Delta^{1/2}$	0.1930	0.3158
0	50	0.02	$\Delta^{3/4}$	0.1284	0.0450
0	100	0.01	$\Delta^{1/2}$	0.1607	0.2281
0	100	0.01	$\Delta^{3/8}$	0.0970	0.0172
0.005	100	0.01	$\Delta^{1/2}$	0.5307	0.7961
0.0025	100	0.01	$\Delta^{1/2}$	0.3406	0.4987

Example 2. Let us consider the system from example 1. Numerical experiments show that

$$\operatorname{ess}\sup_{t\in[t_*,\vartheta]} \|u^h - u_*\|_{L_2(S)} \not\to 0 \quad \text{as} \quad t_* \to t_0, \quad h \to 0.$$

h	m	$\Delta(h)$	$\alpha(h)$	$t_* = \tau_1$	$\|u^h - u_*\|_{L_\infty}$
0	10	0.1	$\Delta^{1/2}$	0.1	0.4835
0	20	0.05	$\Delta^{1/2}$	0.05	0.5490
0	50	0.02	$\Delta^{1/2}$	0.02	0.6071
0	100	0.01	$\Delta^{1/2}$	0.01	0.6364
0	10	0.1	$\Delta^{3/4}$	0.1	0.3095
0	20	0.05	$\Delta^{3/4}$	0.05	0.3727
0	50	0.02	$\Delta^{3/4}$	0.02	0.4412
0	100	0.01	$\Delta^{3/4}$	0.01	0.4835

Example 3. Let us consider the concrete variant of the reconstruction problem. The unknown parameter is boundary disturbance (control). Let the system have the form

$$\begin{cases} y_{tx}(t,x) = y_t(t,x) + y_x(t,x) + y(t,x), & (t,x) \in [0,1]\times[0,1], \\ y(0,x) = 0, & x \in [0,1], \\ y(t,0) = \int\limits_{t_0}^{t} \sin(\tau)d\tau, & t \in [0,1]. \end{cases}$$

The following restriction holds: $\max_{t\in T} \mid v(t) \mid \leq 1$. Let us a priori choose the net $(\tau_i)_{i=\overline{0,m}}$ such that $\Delta(h) = \mid \tau_{i+1} - \tau_i \mid = \frac{1}{m}$, $i = \overline{0, m-1}$. The point $t_* = 0.1$. In the table, the results of the numerical experiment are carried out.

h	m	$\Delta(h)$	$\alpha(h)$	$\|u^h - u_*\|_{L_2}$	$\|u^h - u_*\|_{L_\infty}$
0	10	0.1	$\Delta^{7/8}$	0.0715	0.0899
0	50	0.02	$\Delta^{7/8}$	0.0224	0.0198
0	100	0.01	$\Delta^{7/8}$	0.0138	0.0106
0	10	0.1	$\Delta^{1/2}$	0.0979	0.1687
0	50	0.02	$\Delta^{1/2}$	0.0471	0.0874
0	100	0.01	$\Delta^{1/2}$	0.0332	0.0613
0.01	10	0.1	$\Delta^{1/2}$	0.0927	0.1555
0.004	50	0.02	$\Delta^{1/2}$	0.0470	0.0870
0.001	100	0.01	$\Delta^{1/2}$	0.0332	0.0612

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