

A Time Domain Formulation for Identification in Electromagnetic Dispersion*

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Abstract

We present a time domain approach for the investigation of dispersion mechanisms of a medium in electromagnetic field problems. Maxwell's equations coupled with a generalized electric polarization model are considered. The polarization is given in terms of a convolution of the electric field with an impulse response function. Existence, uniqueness and continuous dependence of solutions on data are presented for a one-dimensional dispersive medium case. Estimation of electromagnetic properties of media is demonstrated via numerical examples. Parameters representing the electromagnetic property of a medium may include the static permittivity, relaxation time, natural frequency, static conductivity, etc. depending on the polarization model chosen.

1 Model Formulation

Microwave images of tissue structures and soils play very important roles in many areas, including clinical and environmental medicine. These microwave images are useful in detection/enhanced treatment of abnormality of human organs and tissue, and detection/ remediation of underground toxic wastes. The electromagnetic properties of a medium are generally characterized by its electric and magnetic polarization mechanisms and its static conductivity. Here we focus on the development of partial differential equation (Maxwell's equations) based identification techniques for

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physical and biological distributed parameter systems, with those for living tissue being a special case. We attempt to estimate the conductivity and parameters which characterize the polarization of media such as living tissue using incident and scattered electromagnetic signals. The functions of permittivity and conductivity are determined by a general polarization mechanism which includes as special cases those governed by an n -th order ordinary differential equation (see [18, 24] for the cases $n = 1, 2$), multiple coupled oscillators, and delay systems. We note that in the literature (which is dominated by frequency domain approaches) complex permittivity and conductivity depending on the frequency of the emitting signal are commonly used as electromagnetic characteristics. Since our approach is in the time domain, we focus directly on the polarization equations and any resulting parameterizations may contain parameters depending on time and space, but not directly on frequency.

Variations of the electromagnetic inverse problem have been studied for at least several decades. A good survey of existing methods is given by Albanese et al. in [1].

However, there is a paucity of literature on studies of electromagnetic inverse problems in the time domain which employ variational formulations. This variational formulation approach has been successfully applied to damped hyperbolic systems in [7, 12, 13] and hybrid systems in [6]. A similar approach with the focus on well posedness and control problems involving Maxwell's equations can be found in [16]. In this reference, Duvaut and Lions proved existence and uniqueness of Maxwell's equations for a three-dimensional inhomogeneous medium with totally reflective boundary. The medium is stable (nonzero static conductivity) and polarization is assumed instantaneous and proportional to the electric field. Efforts by other authors on controllability and stabilization in the context of semigroups and variational formulations can be found in [3, 19, 20, 21, 23].

Our investigations focus on a class of dispersion models (i.e., models for polarization) in the context of Maxwell's equations. We consider the type and rate of attenuation of signals associated with a given dispersion model for biological media.

We begin by considering a three-dimensional inhomogeneous medium with no free charges. The macroscopic equations governing electromagnetic phenomena in time and space are Maxwell's equations

$$\begin{aligned}
 \nabla \cdot \mathbf{D} &= 0 \\
 \nabla \cdot \mathbf{B} &= 0 \\
 \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
 \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_s + \mathbf{J}_c
 \end{aligned} \tag{1}$$

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and along with the constitutive relations

$$\begin{aligned}
 \mathbf{J}_c &= \sigma \mathbf{E} \\
 \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\
 \mathbf{B} &= \mu_0 \mathbf{H} + \mu_0 \mathbf{M}.
 \end{aligned}
 \tag{2}$$

Here \mathbf{E} is the electric field intensity, \mathbf{D} is the electric flux density, \mathbf{H} is the magnetic field intensity, \mathbf{B} is the magnetic flux intensity, \mathbf{J}_s is the source electric current density, \mathbf{J}_c is the conduction current density, \mathbf{P} is the electric polarization, \mathbf{M} is the magnetic polarization, σ is the space dependent static conductivity, ϵ_0 and μ_0 are physical constants representing the permittivity and permeability in vacuum, respectively. In the mks system of units, $\epsilon_0 \approx 8.85 \times 10^{-12} [Fm^{-1}]$ and $\mu_0 = 4\pi \times 10^{-7} [Hm^{-1}]$. The bold faced characters are vectors in Cartesian coordinates.

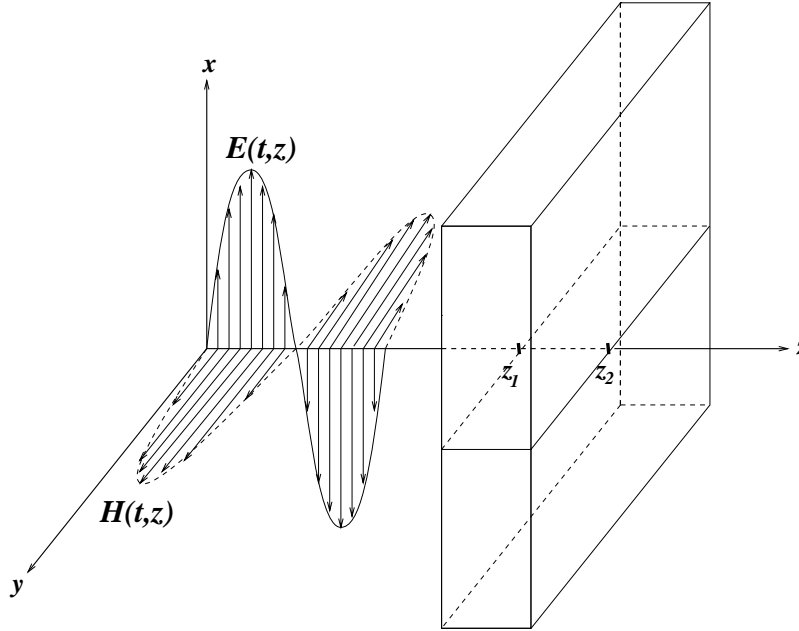


Figure 1: Geometry of Physical Problem

In our initial studies, we concentrate on the propagation of a pulsed plane wave. Assuming the plane wave is uniform in planes parallel to the x - y plane and propagates in z direction (see Figure 1), then the electric and magnetic field intensities are reduced to

$$\mathbf{E} = E(t, z)\hat{i} \quad \text{in the } x \text{ direction}$$

$$\mathbf{H} = H(t, z)\hat{j} \quad \text{in the } y \text{ direction.}$$

For ease in notation, we use henceforth the scalar fields E and H with the understanding that E is polarized in the x or \hat{i} direction and H is polarized in the y or \hat{j} direction. Furthermore, we assume that the magnetic polarization is zero, i.e., $M(t, z) = 0$, which is a good approximation for biological media [2]. With the above assumptions, Maxwell's equations coupled with the constitutive relations yield a second order equation

$$\frac{\partial^2 E}{\partial t^2} + \frac{\sigma}{\epsilon_0} \frac{\partial E}{\partial t} + \frac{1}{\epsilon_0} \frac{\partial^2 P}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} = -\frac{1}{\epsilon_0} \frac{\partial J_s}{\partial t}, \quad (3)$$

$$t > 0, \quad 0 \leq z \leq 1,$$

where $c^2 = 1/\epsilon_0\mu_0$.

To specify boundary conditions we consider the specific geometry of Figure 1 and assume that the one-dimensional inhomogeneous dispersive slab occupies the region $[z_1, z_2]$ such that $0 < z_1 < z_2 < 1$. The medium outside the slab is filled with air in which $\sigma = 0$ and $P = 0$ while in the slab those quantities are nonzero and depend on conductivity and dispersive properties of the medium. We assume totally absorbing boundary conditions for (3) at $z = 0$ and $z = 1$. That is, we take

$$\begin{aligned} \frac{\partial E}{\partial t}(t, 0) - c \frac{\partial E}{\partial z}(t, 0) &= 0, \\ \frac{\partial E}{\partial t}(t, 1) + c \frac{\partial E}{\partial z}(t, 1) &= 0. \end{aligned} \quad (4)$$

While biological media are thought to have magnetic properties of vacuum, dispersion of electromagnetic signals in such heterogeneous media is a complex phenomenon which is usually accounted for in the polarization vector P . In the case of dispersive media (here, we adopt the definition by Stratton [24] which states that a medium is said to be dispersive if the phase velocity in the medium is a function of frequency), one may consider a general representation for the electric polarization described by the n -th order ordinary differential equation

$$\frac{\partial^n P}{\partial t^n} + \sum_{i=1}^n a_i \frac{\partial^{n-i} P}{\partial t^{n-i}} = a_0 E, \quad (5)$$

where the coefficients $\{a_i\}_{i=0}^n$ are space dependent variables. This representation of polarization takes into account the molecular constitution of matter and treats the molecules as dynamical systems possessing natural frequencies. For the case of $n = 0, 1, 2$ the polarization mechanisms have been studied extensively [18, 24]. The case $n = 0$ is called an Ideal medium whereas $n = 1$ and 2 correspond to the so-called Debye medium

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and Lorentz medium, respectively. A medium characterized by multiple Debye models and/or multiple Lorentz models can also be represented by (5) if the coefficients in this equation satisfy certain conditions.

For our investigations we assume a much more general polarization model of the form

$$P(t, z) = \int_0^t g(t - s, z) E(s, z) ds. \quad (6)$$

This model can be found in the research literature on studies of electromagnetic wave propagation in time domain, see for example [14, 22]. It, of course, includes the various *n*-th order models of (5) as special cases whenever the initial polarization is zero. (To readily see this, rewrite (5) as an *n*-vector first order system and use the standard variation-of-parameters representation in terms of the impulse response and the input *E*). The integral formulation (6) will be used in stating precisely general well posedness results while the special cases of the differential formulation are used in some of our computational examples to study specific mechanisms.

The above model (6) also includes as special cases systems with memory, generally referred to in the mathematical literature as time delay systems, systems with hysteresis or hereditary systems [4, 9]. For example, if the polarization rate depends not only on the current polarization but also on the previous values, then one may write

$$\frac{\partial P}{\partial t}(t, z) = \sum_{i=1}^N a_i P(t - r_i, z) + \int_{-r}^0 a(t, s) P(t + s, z) ds + a_0 E(t, z) \quad (7)$$

or, more generally, in terms of Stieltjes measures for memory

$$\frac{\partial P(t, z)}{\partial t} = \int_{-r}^0 P(t + s, z) d_s \eta(t, s) + a_0 E(t, z). \quad (8)$$

It is well known that solutions of (7) or (8) can be written (again assuming no initial polarization) in the form (6) by using variation-of-parameters representations in terms of the fundamental solution $X(t - s)$ and taking $g(t - s) = X(t - s)a_0$ (see [4, 9] for details). It is accepted engineering practice to sometimes approximate the complicated systems (7) or (8)—which are actually infinite dimensional state systems—by finite dimensional high order systems similar to (5) which yield appropriate approximations, e.g., see [5].

2 Well-posedness: Existence, Uniqueness, Continuous Dependence

To treat questions related to existence, uniqueness and continuous dependence of solutions on data, it is convenient to write the system (3), (4), (6) in weak or variational form. We do this in the usual manner using test functions $\phi \in V \equiv H^1(0, 1)$ and a state space $H = L^2(0, 1)$. Multiplying (3) by ϕ , integrating over $z \in [0, 1]$, integrating by parts and using (4) and (6), we obtain

$$\begin{aligned} & \int_0^1 \left(\frac{\partial^2 E}{\partial t^2}(t, z)\phi + \frac{\sigma}{\epsilon_0} \frac{\partial E}{\partial t}(t, z)\phi + c^2 \frac{\partial E}{\partial z}(t, z) \frac{\partial \phi}{\partial z} \right. \\ & \quad \left. + \frac{1}{\epsilon_0} \int_0^t g(t-s, z) \frac{\partial^2 E}{\partial s^2}(s, z) ds \phi \right) dz \\ & \quad + c \frac{\partial E}{\partial t}(t, 1)\phi(1) + c \frac{\partial E}{\partial t}(t, 0)\phi(0) \\ & = \int_0^1 -\frac{1}{\epsilon_0} \frac{\partial J_s}{\partial t}(t, z) \phi dz, \quad t > 0. \end{aligned} \tag{9}$$

With the notation $\cdot = \frac{\partial}{\partial t}$, $' = \frac{\partial}{\partial z}$ and $\langle \phi, \psi \rangle = \int_0^1 \phi \psi dz$, this equation can be written succinctly as

$$\begin{aligned} & \langle \ddot{E}(t), \phi \rangle + \langle \frac{\sigma}{\epsilon_0} \dot{E}, \phi \rangle + \langle \frac{1}{\epsilon_0} \int_0^t g(t-s) \ddot{E}(s) ds, \phi \rangle \\ & + \langle c^2 E'(t), \phi' \rangle = +c \dot{E}(t, 1)\phi(1) + c \dot{E}(t, 0)\phi(0) \langle -\frac{1}{\epsilon_0} \dot{J}_s(t), \phi \rangle \end{aligned} \tag{10}$$

for all $\phi \in V$. We seek solutions satisfying initial conditions $E(0) = \Phi$, $\dot{E}(0) = \Psi$ for $\Phi \in V$, $\Psi \in H$. We note that the functions $z \rightarrow \sigma(z)$ and $z \rightarrow g(t-s, z)$ vanish outside $\tilde{\Omega} = [z_1, z_2]$ for the geometry of Figure 1 (which is our focus here). Further integration by parts in the fourth term of (10) and some tedious calculations enable us to write (10) in the form we use as our basic equation here. It is given by:

$$\begin{aligned} & \langle \ddot{E}(t), \phi \rangle + \langle \gamma \dot{E}(t), \phi \rangle + \langle \beta E(t), \phi \rangle + \langle \int_0^t \alpha(t-s) E(s) ds, \phi \rangle \\ & + \langle c^2 E'(t), \phi' \rangle + c \dot{E}(t, 1)\phi(1) + c \dot{E}(t, 0)\phi(0) = \langle \mathcal{J}(t), \phi \rangle \end{aligned} \tag{11}$$

for all $\phi \in V$. Here

$$\mathcal{J}(t) = -\frac{1}{\epsilon_0} \dot{J}_s,$$

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$$\begin{aligned}\gamma(z) &= \frac{1}{\epsilon_0} \chi_{\bar{\Omega}} [\sigma(z) + g(0, z)], \\ \beta(z) &= \frac{1}{\epsilon_0} \chi_{\bar{\Omega}} \dot{g}(0, z), \\ \alpha(t-s, z) &= \frac{1}{\epsilon_0} \chi_{\bar{\Omega}} \ddot{g}(t-s, z).\end{aligned}$$

The initial conditions are still given by

$$\begin{aligned}E(0, z) &= \Phi(z) \\ z \in \Omega &= (0, 1). \\ \dot{E}(0, z) &= \Psi(z)\end{aligned}\tag{12}$$

Using an approach similar to that of [8],[11, Chapter 4], one can obtain the following well-posedness results for (11), (12) which is the variational form of (3), (4), (6).

Theorem: *Suppose that α, β and $\gamma \in L^\infty(0, 1)$, $\mathcal{J} \in L^2(0, T; L^2(0, 1))$ and $\Phi \in H^1(0, 1)$, $\Psi \in L^2(0, 1)$. Then there exists a unique solution E to (11), (12) with $E \in L^2(0, T; H^1(0, 1))$, $\dot{E} \in L^2(0, T; L^2(0, 1))$, $\ddot{E} \in L^2(0, T; H^1(0, 1)^*)$, $\dot{E}(\cdot, 1) \in L^2(0, T)$, $\dot{E}(\cdot, 0) \in L^2(0, T)$. This solution of (11) is in the usual sense of $L^2(0, T; H^1(0, 1))^* \cong L^2(0, T; H^1(0, 1)^*)$, i.e., (11) is solved in the sense of V^* where $V \equiv H^1(0, 1)$ and the $\langle \cdot, \cdot \rangle$ must be interpreted as the duality product $\langle \cdot, \cdot \rangle_{V^*, V}$ in the first term of (11).*

While this result can be obtained using the approach of [8, 11], the details are somewhat tedious and will be presented elsewhere. Estimates obtained in proving the above theorem can be used to obtain continuous dependence results. We find that the mapping

$$(\Phi, \Psi, \mathcal{J}) \text{ in } H^1(0, 1) \times L^2(0, 1) \times L^2(0, T; H^1(0, 1)^*)$$

to

$$(E, \dot{E}) \text{ in } L^2(0, T; H^1(0, 1)) \times L^2(0, T; L^2(0, 1))$$

is continuous.

We note that the arguments for existence also yield that $t \rightarrow E(t, 0)$ is continuous even though in general one cannot establish the continuity $E \in C(0, T; H^1(0, 1))$ in the usual manner. This will be useful in formulating the inverse problems of the next section.

3 Inverse Problems

One can formulate the identification of electric polarization mechanisms as parameter estimation problems in the time domain and this is the approach

we take here. For given incident (pulse modulated microwave signals – for the importance of such input signals see [2]) and scattered electric fields, the parameter estimation problems consist of finding a set of parameters such that the computed electric field corresponding to the parameters matches in some sense the measured data. In general, one wishes to estimate the parameters α, β, γ of (11) (equivalently σ and g in (10)) or some finite dimensional parameterization of these functions. For example, if one assumes that σ is constant in the slab $\tilde{\Omega} = [z_1, z_2]$ (see Figure 1) and assumes the special case (5) of the polarization model (6), then one might choose $q = (\sigma, a_0, a_1, \dots, a_n)$ as the vector of unknown parameters to be estimated using observed data at $z = 0$ from the reflected electric field. More generally, let q be a vector parameterization for α, β, γ with values ranging over some admissible set Q . We assume that we have measurements $\tilde{E} = \{\tilde{E}_i\}$ of the electric field E at $z = 0$ for times $\{t_i\}$. Then we consider the least squares estimation problem of minimizing over $q \in Q$ the criterion

$$J(\tilde{E}, q) = \sum_{i=1}^{N_t} |E(t_i, 0; q) - \tilde{E}_i|^2 \quad (13)$$

where $E(t_i, 0; q)$ are solutions corresponding to q of (11), (12) evaluated at $(t, z) = (t_i, 0)$.

The minimization in our parameter estimation problems involves an infinite dimensional state space and approximations must be made to obtain solutions. We thus consider Galerkin type approximations in the context of the variational formulation (11). Solving the associated approximate estimation problems, we obtain a sequence of estimates $\{\tilde{q}^N\}$ where N is an index of approximation - see [11, Chapter 5] for a general description and convergence results for similar problems. For the problems considered here, parameter estimate convergence and continuous dependence (with respect to the observations $\{\tilde{E}_i\}$) results can be given under certain assumptions but involve nontrivial extensions of arguments such as those found in [8, 11]. These will be given elsewhere.

One can also give more general convergence results which allow function space approximation of the parameter set for α, β, γ directly. These ideas provide a sound theoretical basis for the reconstruction of conductivity and the general kernel in the polarization equation (6) and will also be discussed fully elsewhere.

It is well known that evaluation of electric and magnetic fields in the microwave range is computationally expensive in the time domain. To minimize the number of function evaluations, an iterative Trust Region algorithm [15] was employed in solving the above minimization problem in a series of computational examples we used to test our ideas. In addition to good convergence properties, the algorithm has the potential to provide a global minimum for our problem. We combined the trust region algorithm

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with a piecewise linear spline approximation scheme for the states in (11) to produce the numerical results reported in the next section.

4 Numerical Results

In some preliminary computational examples, we have tested the ideas presented above with a special case of (6), the so-called Debye model for polarization. Specifically, we replaced D given in (2) by $D = \epsilon(z)E + P$ where $\epsilon(z) = \epsilon_0$ for $z \notin \tilde{\Omega}$, $\epsilon(z) = \epsilon_0\epsilon_\infty$ for $z \in \tilde{\Omega}$ and P is given by

$$\tau\dot{P} + P = \epsilon_0(\epsilon_s - \epsilon_\infty)E, \quad (14)$$

where ϵ_s and ϵ_∞ are the static relative permittivity and high frequency relative permittivity, respectively, and τ is the relaxation time. This definition for D allows for some instantaneous polarization in $\tilde{\Omega}$ and maintains parameterization equivalent to that found in the frequency domain literature. The theory presented above readily treats this modification.

We first carried out a series of simulations for the electric field corresponding to a time “windowed” point source input at the boundary point $z = 0$ given by

$$\mathcal{J}(t, z) = -\delta(z)\chi_{[0, t_f]} \sin \omega t. \quad (15)$$

We present below a series of figures depicting the field at various times for one such example. These calculations were performed using piecewise linear splines in the state approximation with an approximation index of $N = 450$. Time stepping was discretized with $\Delta t = 33. \times 10^{-5}$ nsec. The input is one pulse with a pulse duration of 3.33 nsec. The carrier frequency $f = \omega/2\pi$ was given by $f = 1.8 \times 10^9$ Hz and parameters $\epsilon_s = 35$, $\epsilon_\infty = 5$, $\tau = 1 \times 10^{-11}$, $c = 3.0 \times 10^8$ were used in the simulation depicted. The slab was located in $\tilde{\Omega} = [z_1, z_2] = [.33, .89]$ and Figure 2 depicts the incoming pulse before it has reached the slab. Figure 3 is a plot of $E(t, z)$ at $t = 5$ nsec. and the field at this time consists of the partially reflected (at the $z = z_1$ interface) field in free space and the partially transmitted field inside the slab. In Figure 4 we see the initiation of precursor formation (the familiar Brillouin precursors) in the slab while Figure 5 reveals a clearer precursor pair formulation. Numerous simulations similar to these were carried out and compared to other methods (e.g., Fourier series solutions) to verify the accuracy of our computational packages before we began testing of the inverse problem algorithms.

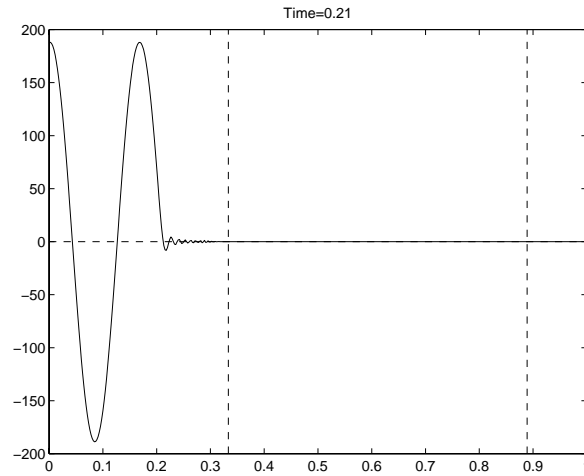


Figure 2: Electric Field E at Time $t = .7$ nsec.

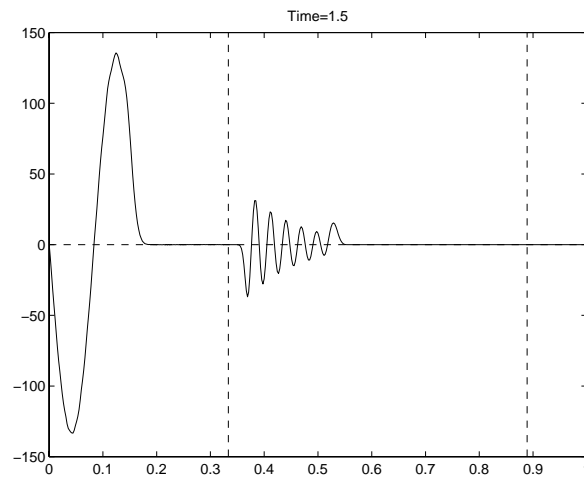


Figure 3: Electric Field E at Time $t = 5$ nsec.

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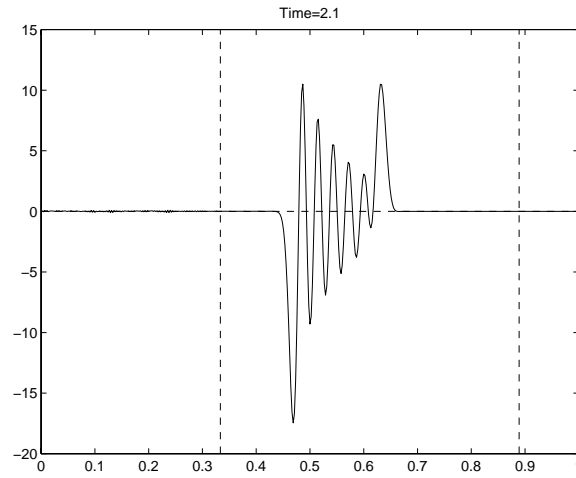


Figure 4: Electric Field E at Time $t = 7$ nsec.

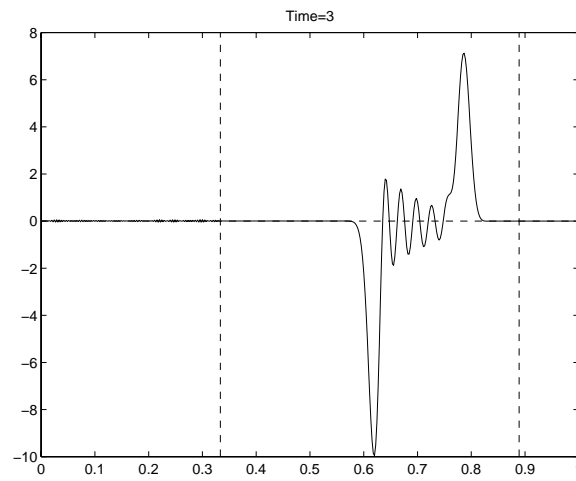


Figure 5: Electric Field E at Time $t = 10$ nsec.

To test our inverse algorithms, we used simulations as described above to generate “simulated data” - see [10] for a general description of this procedure for testing inverse algorithms - to which “relative noise” was added before it was employed as “observations” in (13). We report here briefly on one such test. The simulated data was generated with “true” values (“water” - see [2]): $\sigma^* = 10^{-5}$, $\tau^* = 8.1 \times 10^{-12}$, $\epsilon_s^* = 80.1$, $\epsilon_\infty^* = 5.5$ with $f = 1.8 \times 10^9$ Hz as the carrier frequency in (15) and pulse duration of $t_f = 3.3$ nsec. “Data” was produced over the time interval $[0, T]$ with $T = 10$ nsec and sampling at $t_i = i\Delta t$, $i = 1, \dots, 100$, $\Delta t = .1$ nsec. Several different levels of relative noise were added to test performance of the algorithm in the presence of increasing noise.

For no noise in the data, the algorithm essentially converged from initial estimates $\sigma^0 = 1.5 \times 10^{-5}$, $\tau^0 = 10 \times 10^{-12}$, $\epsilon_s^0 = 73$, $\epsilon_\infty^0 = 6$, to the true values $\bar{\tau} \approx \tau^*$, $\bar{\epsilon}_s \approx \epsilon_s^*$, $\bar{\epsilon}_\infty \approx \epsilon_\infty^*$ with the exception for $\bar{\sigma} = 0$. It was found in this case as well as in other tests that the data was such that the algorithm either converged to $\bar{\sigma} = 0$ with convergence to $\bar{\epsilon}_\infty \approx \epsilon_\infty^*$ or to $\bar{\sigma} \approx \sigma^*$ with $\bar{\epsilon}_\infty = 0$. That is, sensitivity of the solutions to σ and ϵ_∞ in this range of values tested was such that it was not possible to identify σ and ϵ_∞ independently. Results for several levels of noise are presented in the table below. As expected, the performance deteriorated with increasing noise levels.

Noise Level	Estimated Values			
	$\bar{\sigma}$	$\bar{\tau}$	$\bar{\epsilon}_s$	$\bar{\epsilon}_\infty$
2%	1.69×10^{-5}	8.43×10^{-12}	79.88	0
3%	9.29×10^{-5}	8.9×10^{-12}	79.8	0
5%	9.47×10^{-5}	9.8×10^{-12}	79.64	0

Table 1: Estimated Parameters in the Presence of Noise

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