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Weak Attractor for Damped Abstract Nonlinear Hyperbolic Systems*

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Abstract

This paper is concerned with a class of damped abstract nonlinear hyperbolic systems that arise in the study of certain smart material structures. The present work can be considered as a continuation of the work of H.T Banks, D.S.Gilliam and V.I. Shubov in which the existence and uniqueness of weak solutions for this class of systems was recently established. In particular, with the addition of one technical assumption, we prove the existence of a weak dynamical system, a weak compact global attractor, the existence of a global Lyapunov function and make some statements concerning the asymptotic behavior of solutions for these systems. We note that, even though the existence of a strong dynamical system for this class has not been proven, it would not imply the existence of a weak dynamical system which is typically harder to characterize. The advantage of a weak dynamical system is that it is often easier to prove the existence of the weak compact attractor.

Key words: abstract hyperbolic systems, weak dynamical system, weak compact attractor

AMS Subject Classifications: 35B40

1 Introduction

In this work we consider a class of abstract nonlinear damped hyperbolic systems evolving in a complex separable Hilbert space. This class of nonlinear systems, first studied in the recent papers [3, 4], arise as dynamical models for smart material structures, or more precisely, for elastomers. These are rubber based products with a variety of applications in modern

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material sciences. One such application is the development of active and passive vibration devices. The nonlinearity in these systems comes from a nonlinear relationship between stress and strain that these elastomers are known to exhibit. The study of these systems is also important for the development of computational methodologies for the identification and control of smart material structures. For a detailed discussion see [4, 6].

The class of systems can be described by damped nonlinear hyperbolic equations of the form

$$w_{tt} + \mathcal{A}_1 w + \mathcal{A}_2 w_t + \mathcal{N}^* g(\mathcal{N} w) = f(t) \tag{1.1}$$

$$w(0) = \varphi_0 \tag{1.2}$$

$$w_t(0) = \varphi_1 \tag{1.3}$$

in a separable Hilbert space $\mathcal{H}$. The equation is actually to hold in the sense of a larger space $\mathcal{V}^*$ the dual of $\mathcal{V}$. Here $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{N}$ are unbounded linear operators, g is a continuous nonlinear operator in $\mathcal{H}$ and f is a, possibly distributional, external forcing term. Precise conditions on the spaces and operators involved are given in Section 3.

The global in time existence and uniqueness of the weak solution for the problem (1.1)-(1.3) was established in [3, 4]. As a continuation of this work, we have adopted all the notation introduced in [4]. In [4] the authors were primarily interested in proving existence and uniqueness of solutions, therefore they did not treat the question of whether the systems define a weak or strong dynamical system, nor did they consider the asymptotic behavior of solutions. In the present work we examine some of these issues and show, in particular, that the problems define a weak dynamical system and prove the existence of a weak compact attractor. We note that it is usually easier to prove that the system defines a strong dynamical system in the state space, but theorems guaranteeing the existence of an attractor usually require proving that the trajectories of the system are precompact in the state space [8, 10, 13]. This property is essential to insure the existence of a limit point for bounded trajectories. However, this precompactness is often difficult to establish. One alternative is to use the weak topology on the state space, in which case boundedness suffices for precompactness of the trajectories. This idea was introduced and developed in several papers [9, 7, 12, 1, 2]. Once the existence of a weak dynamical system is shown, it is sufficient to prove that the system is bounded and point dissipative to insure the existence of a weak compact attractor.

The paper is organized in the following manner. In Section 2 we have included the necessary definitions and a theorem from [9] which give the extensions of the usual concepts and statements to weak dynamical systems. In Section 3 we give all necessary assumptions about the problem (1.1)-(1.3). In Section 4 (as we mentioned earlier) we prove that (1.1)-(1.3) gives rise to a weak dynamical system. In Section 5 we state and prove our main result concerning the existence of a weak compact attractor. Section 6 is devoted to showing the existence of a Lyapunov function and describing the asymptotic behavior of weak solutions.

2 Weak Dynamical Systems and Weak Compact Attractor

To examine the behavior of weak solutions we introduce the notion of a weak dynamical system [9, 12].

Definition 2.1 *Let X be a reflexive Banach space, and let $\tilde{X}$ denote the space X endowed with the weak topology. A* **weak dynamical system** *on $\tilde{X}$ is a function T: $\mathbb{R}^+ \times \tilde{X} \to \tilde{X}$ with the following properties:*

(i) $T(0)x = x$ *for all* $x \in X$.

(ii) $T(t+\tau)x = T(t)T(\tau)x$ *for all* $t, \tau \in \mathbb{R}^+$, $x \in X$.

(iii) $T(.)x : t \to T(t)x$ *is continuous from* $\mathbb{R}^+$ *into* $\tilde{X}$ *for fixed* $x \in X$.

(iv) $T(t) :\ x \to T(t)x$ *is weakly sequentially continuous for fixed* $t \in \mathbb{R}^+$ *(i.e. if* $x_n \to x$ *weakly in X, then* $T(t)x_n \to T(t)x$ *weakly in X).*

In Section 3 we show that (1.1)-(1.3) gives rise to a weak dynamical system on $\mathcal{V} \times \mathcal{H}$.

Definition 2.2 *A set $K \subset \tilde{X}$ is a* **weak compact attractor** *for $T(t)$ in $\tilde{X}$, if it is maximal, weak compact, invariant and weakly attracts the bounded sets of $\tilde{X}$; i.e. for any bounded set $B \subset X$ and any $\varepsilon > 0$, there is a $t_0 = t_0(\varepsilon, B, K)$ such that $T(t)B \subset \tilde{N}_\varepsilon(K)$ for $t \geq t_0$, where $\tilde{N}_\varepsilon(K)$ is a weak ε-neighborhood of K.*

Definition 2.3 *The weak dynamical system T(t) is said to be* **weak point dissipative** *if there is a bounded set $K \subset \tilde{X}$, which weakly attracts the points of X.*

Definition 2.4 *A set K is said to be* **weakly stable** *if, for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $T(t)\tilde{N}_\delta(K) \subset \tilde{N}_\varepsilon(K)$, for all $t \geq 0$.*

Theorem 2.1 *Let X be a separable reflexive Banach space, $T(t) : \mathbb{R}^+ \times X \to X$ be a weak dynamical system with $T(t,.)$ weak point dissipative. Also assume that $\gamma^+(B)$ (the set of positive semi-trajectories starting from B) is a bounded subset of X. Then there exists a weak compact attractor which is weakly stable and weakly connected.*

3 Formulation of Problem

Using the same notations as in [4], we assume that there is a sequence of separable Hilbert spaces $\mathcal{V}$, $\mathcal{V}_2$, $\mathcal{H}$, $\mathcal{V}^*$, $\mathcal{V}_2^*$ that form a Gelfand quintuple [5, 14]:

$$\mathcal{V} \hookrightarrow \mathcal{V}_2 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}_2^* \hookrightarrow \mathcal{V}^*.$$

The embedding $\mathcal{V} \hookrightarrow \mathcal{V}_2$ is dense and continuous with $\|\varphi\|_{\mathcal{V}_2} \leq c\|\varphi\|_{\mathcal{V}}$ for $\varphi \in \mathcal{V}$ and $\mathcal{V}_2 \hookrightarrow \mathcal{H}$ is a dense compact embedding with $\|\varphi\| \leq \tilde{c}\|\varphi\|_{\mathcal{V}_2}$. We denote by $\langle\ ,\ \rangle_{\mathcal{V}^*,\mathcal{V}}$, etc., the usual duality products [14]. These duality products are the extensions by continuity of the inner product in $\mathcal{H}$ which is denoted by $\langle\ ,\ \rangle$. The norm in $\mathcal{H}$ will be denoted by $\|\cdot\|$ while those in $\mathcal{V}$, $\mathcal{V}_2$ etc. will have an appropriate subscript. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined in terms of their sesquilinear forms $\sigma_1 : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ and $\sigma_2 : \mathcal{V}_2 \times \mathcal{V}_2 \to \mathbb{C}$. That is, $\mathcal{A}_1 \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*), \mathcal{A}_2 \in \mathcal{L}(\mathcal{V}_2, \mathcal{V}_2^*)$ and $\langle \mathcal{A}_1\varphi, \psi\rangle_{\mathcal{V}^*,\mathcal{V}} = \sigma_1(\varphi, \psi), \langle \mathcal{A}_2\varphi, \psi\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} = \sigma_2(\varphi, \psi)$.

Let $\mathcal{L}_T$ denote the space of functions $w : [0,T] \to \mathcal{H}$ such that

$$w \in C_W([0,T], \mathcal{V}_2) \cap L^\infty([0,T], \mathcal{V})$$

(W means weak continuity), and

$$w_t \in C_W([0,T], \mathcal{H}) \cap L^2([0,T], \mathcal{V}_2),$$

where the time derivative w_t is understood in the sense of distributions with values in a Hilbert Space (see, e.g., [11]). The space $\mathcal{L}_T$ is equipped with the norm

$$\|w\|_{\mathcal{L}_T} = \operatorname*{ess\ sup}_{t\in[0,T]} \left(\|w_t(t)\| + \|w(t)\|_{\mathcal{V}}\right) + \left(\int_0^T \|w_t(t)\|_{\mathcal{V}_2}^2 dt\right)^{1/2}. \quad (3.1)$$

Definition 3.1 *We say that $w \in \mathcal{L}_T$ is a* **weak solution** *of the problem (1.1) – (1.3) if it satisfies the equation:*

$$\int_0^t \Big[-\langle w_\tau(\tau), \eta_\tau(\tau)\rangle + \sigma_1\left(w(\tau), \eta(\tau)\right) + \sigma_2\left(w_\tau(\tau), \eta(\tau)\right) +$$

$$+\langle g\left(\mathcal{N}w(\tau)\right), \mathcal{N}\eta(\tau)\rangle\Big] d\tau + \langle w_t(t), \eta(t)\rangle =$$

$$= \langle \varphi_1, \eta(0)\rangle + \int_0^t \langle f(\tau), \eta(\tau)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} d\tau, \quad (3.2)$$

for any $t \in [0,T]$ and any $\eta \in \mathcal{L}_T$, as well as the initial condition

$$w(0) = \varphi_0. \quad (3.3)$$

Equivalently,

$$\langle w_{tt}, \eta\rangle_{\mathcal{V}^*,\mathcal{V}} + \sigma_1(w,\eta) + \sigma_2(w_t,\eta) + \langle g(\mathcal{N}w), \mathcal{N}\eta\rangle = \langle f, \eta\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \tag{3.4}$$

is satisfied for all $\eta \in \mathcal{L}_T$ *and for almost all* $t \in [0,T]$.

We make the following assumptions (these assumptions are the same as in [4] except that in A5) we require the real part of σ_2 to be strictly coercive and in A6), since we are interested in the existence of a weak attractor, we assume that f does not depend on t; moreover, we introduce one additional assumption A12)):

A1) The form σ_1 is a Hermitian sesquilinear form: for $\varphi, \psi \in \mathcal{V}$

$$\sigma_1(\varphi,\psi) = \overline{\sigma_1(\psi,\varphi)}. \tag{3.5}$$

A2) The form σ_1 is $\mathcal{V}$ bounded: for $\varphi, \psi \in \mathcal{V}$

$$|\sigma_1(\varphi,\psi)| \leq c_1 \|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}. \tag{3.6}$$

A3) The form σ_1 is strictly coercive on $\mathcal{V}$: for $\varphi \in \mathcal{V}$

$$\operatorname{Re}\sigma_1(\varphi,\varphi) = \sigma_1(\varphi,\varphi) \geq k_1 \|\varphi\|_{\mathcal{V}}^2, \quad k_1 > 0. \tag{3.7}$$

A4) The form σ_2 is bounded on $\mathcal{V}_2$: for $\varphi, \psi \in \mathcal{V}_2$

$$|\sigma_2(\varphi,\psi)| \leq c_2 \|\varphi\|_{\mathcal{V}_2} \|\psi\|_{\mathcal{V}_2}. \tag{3.8}$$

A5) The real part of σ_2 is strictly coercive and symmetric on $\mathcal{V}_2$:

$$\operatorname{Re}\sigma_2(\varphi,\varphi) \geq k_2 \|\varphi\|_{\mathcal{V}_2}^2, \quad k_2 > 0 \tag{3.9}$$

$$\operatorname{Re}\sigma_2(\varphi,\psi) = \operatorname{Re}\sigma_2(\psi,\varphi), \text{ for any } \varphi, \psi \in \mathcal{V}_2. \tag{3.10}$$

A6) The forcing term f is time-independent, $f \in \mathcal{V}_2^*$.

A7) The operator $\mathcal{N}$ satisfies

$$\mathcal{N} \in \mathcal{L}(\mathcal{V}_2, \mathcal{H}) \text{ with } \|\mathcal{N}\varphi\| \leq \sqrt{\tilde{k}}\, \|\varphi\|_{\mathcal{V}_2} \tag{3.11}$$

and the range of $\mathcal{N}$ on $\mathcal{V}$ is dense in $\mathcal{H}$.

Note that (3.11) together with $\mathcal{V}_2 \hookrightarrow \mathcal{H}$ implies

$$\mathcal{N} \in \mathcal{L}(\mathcal{V}, \mathcal{H}) \text{ with } \|\mathcal{N}\varphi\| \leq \sqrt{k}\, \|\varphi\|_{\mathcal{V}} \tag{3.12}$$

with $k = \tilde{c}^2 \tilde{k}$.

A8) The nonlinear function $g : \mathcal{H} \to \mathcal{H}$ is a continuous nonlinear mapping of real gradient (or potential) type. This means that there exists a continuous Frechet-differentiable nonlinear functional $G : \mathcal{H} \to I\!R^1$, whose Frechet derivative $G'(\varphi) \in \mathcal{L}(\mathcal{H}, R^1)$ at any $\varphi \in \mathcal{H}$ can be represented in the form

$$G'(\varphi)\psi = \mathrm{Re}\langle g(\varphi), \psi\rangle \quad \text{for any } \psi \in \mathcal{H}. \tag{3.13}$$

We also require that there are constants $C_1,\ C_2,\ C_3$ and $\varepsilon > 0$ such that

$$-\frac{1}{2}k^{-1}(k_1 - \varepsilon)\|\varphi\|^2 - C_1 \le G(\varphi) \le C_2\|\varphi\|^2 + C_3, \tag{3.14}$$

where k is from (3.12) and k_1 from (3.7).

A9) The nonlinear function g also satisfies

$$\|g(\varphi)\| \le \widetilde{C}_1\|\varphi\| + \widetilde{C}_2, \quad \varphi \in \mathcal{H}, \tag{3.15}$$

for some constants $\tilde{C}_1,\ \tilde{C}_2$.

An additional condition is necessary for uniqueness of solutions.

A10) For any $\varphi \in \mathcal{H}$ the Frechet derivative of g exists and satisfies

$$g'(\varphi) \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \text{ with } \|g'(\varphi)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le \widetilde{C}_3. \tag{3.16}$$

A11) We assume that for any $u, v \in \mathcal{L}_T$, the following inequality is satisfied for any $t \in [0, T]$:

$$\begin{aligned} &\int_0^t \Big\{ \mathrm{Re}\langle g(\mathcal{N}u(\tau)) - g(\mathcal{N}v(\tau)), \mathcal{N}u(\tau) - \mathcal{N}v(\tau)\rangle \\ &+k_1k^{-1}\|\mathcal{N}u(\tau) - \mathcal{N}v(\tau)\|^2\Big\}\, dt \\ &+a\left(\left(\int_0^t \|u(\tau) - v(\tau)\|^2\, dt\right)^{1/2}\right) \ge 0, \end{aligned} \tag{3.17}$$

where $a(\xi) \ge 0$ is a continuous function in $\xi \ge 0$ such that

i) $a(0) = 0$,

ii) there exists a first derivative such that $a'(0) = 0$.

Note that (3.17) is satisfied if, for example,

$$\mathrm{Re}\langle g(\varphi) - g(\psi), \varphi - \psi\rangle + k_1k^{-1}\|\varphi - \psi\|^2 \ge 0 \tag{3.18}$$

for any $\varphi, \psi \in \mathcal{H}$, where k and k_1 are the constants in (3.12) and (3.7).

A12) The embedding $\mathcal{V} \hookrightarrow \mathcal{V}_2$ is compact.

4 The Weak Dynamical System

At this point we recall the following result from [4].

Theorem 4.1 *Under conditions A1)-A11) the system (2.1) has a unique weak solution* $w \in \mathcal{L}_T$ *for every initial condition* $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in \mathcal{V} \times \mathcal{H}$. *The weak solution satisfies*

$$\langle w_{tt}, \eta \rangle_{\mathcal{V}^*,\mathcal{V}} + \sigma_1(w, \eta) + \sigma_2(w_t, \eta) + \langle g(\mathcal{N}w), \mathcal{N}\eta \rangle = \langle f, \eta \rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \qquad (4.1)$$

for all $\eta \in \mathcal{L}_T$ *and for almost all* $t \in [0, T]$. *Moreover,* $w \in C_W([0, T], \mathcal{V})$.

Using this Theorem we can define the solution operator $S_t : \mathcal{V} \times \mathcal{H} \to \mathcal{V} \times \mathcal{H}$ by

$$S_t \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix},$$

where w is the weak solution of (1.1) corresponding to the initial condition $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$. Now we can prove the following theorem:

Theorem 4.2 *If conditions A1)-A12) are satisfied, then* $\{S_t, t \geq 0, \mathcal{V} \times \mathcal{H}\}$ *is a weak dynamical system in the sense of Definition 2.1.*

Proof: It is clear that $\{S_t, t \geq 0, \mathcal{V} \times \mathcal{H}\}$ satisfies the semigroup properties (i)-(ii) (since by A6) f does not depend on t). Weak continuity in t (i.e. property (iii)) also comes immediately from Theorem 4.1. So the main difficulty lies in establishing property (iv), the weak continuous dependence on initial conditions. Suppose that $\begin{pmatrix} \varphi_0^n \\ \varphi_1^n \end{pmatrix} \in \mathcal{V} \times \mathcal{H}$ and $\begin{pmatrix} w^n(t) \\ w_t^n(t) \end{pmatrix}$ are the corresponding weak solutions of (1.1). We want to show that if $\begin{pmatrix} \varphi_0^n \\ \varphi_1^n \end{pmatrix} \to \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ weakly in $\mathcal{V} \times \mathcal{H}$, then $\begin{pmatrix} w^n(t) \\ w_t^n(t) \end{pmatrix} \to \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix}$ weakly in $\mathcal{V} \times \mathcal{H}$, where $\begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix}$ is the weak solution corresponding to $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$. To achieve this, we closely follow the steps of the proof of Theorem 6.1. in [4]. There it is shown, that for given initial condition $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$ the Galerkin approximations $w^N(t)$, $w_t^N(t)$ satisfy the following inequality:

$$\|w_t^N(t)\|^2 + \varepsilon \|w^N(t)\|_{\mathcal{V}}^2 + k_2 \int_0^T \|w_\tau^N(\tau)\|_{\mathcal{V}_2}^2 d\tau \leq \tilde{C}(\psi_0, \psi_1, f, T). \qquad (4.2)$$

Using the weak convergences $w^N(t) \rightharpoonup w(t)$ in $\mathcal{V}$, $w_t^N(t) \rightharpoonup w_t(t)$ in $\mathcal{H}$ and $w_t^N \rightharpoonup w_t$ in $L^2([0,T],\mathcal{V}_2)$ (see Lemma 5.1. in [4]), and the lower semicontinuity of norms we get:

$$\|w_t(t)\|^2 + \varepsilon\|w(t)\|_{\mathcal{V}}^2 + k_2\int_0^T \|w_\tau(\tau)\|_{\mathcal{V}_2}^2 d\tau \leq$$

$$\underline{\lim}_{N\to\infty}\|w_t^N(t)\|^2 + \varepsilon\|w^N(t)\|_{\mathcal{V}}^2 + k_2\int_0^T \|w_\tau^N(\tau)\|_{\mathcal{V}_2}^2 d\tau \leq$$

$$\tilde{C}(\psi_0,\psi_1,f,T).$$

Since weak convergence of $\begin{pmatrix} \varphi_0^n \\ \varphi_1^n \end{pmatrix}$ implies norm boundedness, there exist K_1, K_2 such that $\|\varphi_0^n\|_{\mathcal{V}} \leq K_1$ and $\|\varphi_1^n\| \leq K_2$. Thus, we can conclude that there exists $C > 0$ depending on K_1, K_2, f, T such that

$$\|w_t^n(t)\|^2 + \varepsilon\|w^n(t)\|_{\mathcal{V}}^2 + k_2\int_0^T \|w_\tau^n(\tau)\|_{\mathcal{V}_2}^2 d\tau \leq C(K_1,K_2,f,T) \tag{4.3}$$

for every $n \geq 1$. It follows from (4.3) that $\{w^n\}$ is bounded in $L^\infty([0,T],\mathcal{V}) \subset L^2([0,T],\mathcal{V})$ and that $\{w_t^n\}$ is bounded in $L^2([0,T],\mathcal{V}_2)$. So there exists a subsequence such that

$$w^n \longrightarrow w \text{ weakly in } L^2([0,T],\mathcal{V}) \tag{4.4}$$

$$w_t^n \longrightarrow w_t \text{ weakly in } L^2([0,T]),\mathcal{V}_2). \tag{4.5}$$

Now we can establish the following convergences:

a)
$$w^n(t) \to w(t) \text{ weakly in } \mathcal{V}_2 \tag{4.6}$$
uniformly in $t \in [0,T]$;

b)
$$w_t^n(t) \to w_t(t) \text{ weakly in } \mathcal{H} \tag{4.7}$$
uniformly in $t \in [0,T]$;

c)
$$w_t^n \to w_t \text{ strongly in } L^2([0,T],\mathcal{H}); \tag{4.8}$$

d) there exists $h \in L^2([0,T],\mathcal{H})$ such that
$$g(\mathcal{N}w^n) \to h \text{ weakly in } L^2([0,T],\mathcal{H}); \tag{4.9}$$

Assuming A12), i.e. $\mathcal{V} \hookrightarrow \mathcal{V}_2$ is compact, (4.6)-(4.9) can be obtained by the same arguments as the similar convergences in [4] for Galerkin approximations, since in that case $\{w^n\} \subset C_W([0,T],\mathcal{V})$ implies that $\{w^n\} \subset C([0,T],\mathcal{V}_2)$. However, (4.6)-(4.9) can also be established without the assumption A12), but then we have to utilize a different version of the Ascoli-Arzela Theorem (see [2] p.253). Once the above convergences are obtained, we can again proceed similarly as in [4] and complete the proof. The only place where the additional assumption A12) plays a crucial role (and therefore cannot be omitted) is the following: in the calculations showing that $\begin{pmatrix} w \\ w_t \end{pmatrix}$ is a weak solution corresponding to $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ we need that

$$w^n(0) = \varphi_0^n \to \varphi_0 \text{ strongly in } \mathcal{V}_2. \tag{4.10}$$

(This is needed for example to get that $\sigma_2(w^n(0), w^n(0)) \to \sigma_2(\varphi_0, \varphi_0)$.) Since originally $\varphi_0^n \to \varphi_0$ only weakly in $\mathcal{V}$, (4.10) does not necessarily take place unless we assume that the embedding $\mathcal{V} \hookrightarrow \mathcal{V}_2$ is compact.

Q.E.D.

5 Existence of a Weak Attractor

In this section we prove the following theorem:

Theorem 5.1 *Under conditions A1)-A12) the weak dynamical system $\{S_t, t \geq 0, \mathcal{V} \times \mathcal{H}\}$ possesses a weak compact attractor $\mathcal{K}$ which is weakly stable and weakly connected.*

Proof: We prove that weak solutions satisfy an inequality which guarantees the boundedness of $\gamma^+(B)$ when $B \subset \mathcal{V} \times \mathcal{H}$ is bounded and the weak point dissipativeness of the weak dynamical system $\{S_t, t \geq 0, \mathcal{V} \times \mathcal{H}\}$. Then by Theorem 2.1 the proof is complete. We proceed formally. Choosing $\eta = w_t$ in (4.1) and taking real part we obtain:

$$\frac{d}{dt}\left\{\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\sigma_1(w,w) + G(\mathcal{N}w)\right\} + \operatorname{Re}\sigma_2(w_t, w_t) = \operatorname{Re}\langle f, w_t\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \tag{5.1}$$

where we have used the fact that due to (3.13) we have

$$\frac{d}{dt}G(\mathcal{N}w) = \operatorname{Re}\langle g(\mathcal{N}w), \mathcal{N}w_t\rangle.$$

Using A5) and multiplying by 2 we get:

$$\frac{d}{dt}\left\{\|w_t\|^2 + \sigma_1(w,w) + 2G(\mathcal{N}w)\right\} + 2k_2\|w_t\|^2_{\mathcal{V}_2} \leq 2\delta\|w_t\|^2_{\mathcal{V}_2} + \frac{1}{2\delta}\|f\|^2_{\mathcal{V}_2^*}. \tag{5.2}$$

Choosing δ such that $k_2-\delta>0$ we get $2(k_2-\delta)\|w_t\|^2_{\mathcal{V}_2}\geq\frac{2(k_2-\delta)}{\tilde{c}}\|w_t\|^2$. Let $l=\frac{2(k_2-\delta)}{\tilde{c}}$. Multiplying by e^{lt} and then integrating from 0 to t we obtain:

$$e^{lt}\|w_t(t)\|^2-\|w_t(0)\|^2+\int_0^t e^{ls}\frac{d}{ds}\sigma_1(w(s),w(s))ds$$

$$+\int_0^t 2e^{ls}\frac{d}{ds}(G\mathcal{N}w(s))ds\leq\frac{1}{2\delta}\int_0^t e^{ls}\|f\|^2_{\mathcal{V}_2^*}ds.$$

Using integration by parts we have:

$$e^{lt}\|w_t(t)\|^2-\|\varphi_1\|^2+e^{lt}\sigma_1(w(t),w(t))-\sigma_1(w(0),w(0))$$

$$-\int_0^t le^{ls}\sigma_1(w(s),w(s))ds+2e^{lt}G(\mathcal{N}w(t))-2G(\mathcal{N}w(0))$$

$$-\int_0^t 2le^{ls}G(\mathcal{N}w(s))ds\leq\frac{1}{2\delta}\int_0^t e^{ls}\|f\|^2_{\mathcal{V}_2^*}ds, \tag{5.3}$$

which gives (using A3), A2)):

$$\|w_t(t)\|^2+k_1\|w(t)\|^2_{\mathcal{V}}+2G(\mathcal{N}w(t))\leq$$

$$e^{-lt}\left(\|\varphi_1\|^2+c_1\|\varphi_0\|^2_{\mathcal{V}}+2G(\mathcal{N}w(0))\right)$$

$$+l\int_0^t e^{l(s-t)}\sigma_1(w(s),w(s))ds+2l\int_0^t e^{l(s-t)}G(\mathcal{N}w(s))ds$$

$$+\frac{1}{2\delta}\int_0^t e^{l(s-t)}\|f\|^2_{\mathcal{V}_2^*}ds. \tag{5.4}$$

By A8), A7), A3) we obtain:

$$\|w_t(t)\|^2+k_1\|w(t)\|^2_{\mathcal{V}}+2\left[-\frac{1}{2}k^{-1}(k_1-\varepsilon)k\|w(t)\|^2_{\mathcal{V}}-C_1\right]\leq$$

$$e^{-lt}\left[\|\varphi_1\|^2+c_1\|\varphi_0\|^2_{\mathcal{V}}+2C_2k\|\varphi_0\|^2_{\mathcal{V}}+2C_3\right]+lc_1\int_0^t e^{l(s-t)}\|w(s)\|^2_{\mathcal{V}}ds$$

$$+2l\int_0^t e^{l(s-t)}(C_2k\|w(s)\|^2_{\mathcal{V}}+C_3)ds$$

$$+\frac{1}{2\delta}\int_0^t e^{l(s-t)}\|f\|^2_{\mathcal{V}_2^*}ds. \tag{5.5}$$

This gives

$$\|w_t(t)\|^2+\varepsilon\|w(t)\|^2_{\mathcal{V}}\leq e^{-lt}K+\left(2lC_3+\frac{\|f\|^2_{\mathcal{V}_2^*}}{2\delta}\right)\int_0^t e^{l(s-t)}ds$$

$$+2C_1 + L\int_0^t e^{l(s-t)}(\|w(s)\|_{\mathcal{V}}^2 + \|w_s(s)\|^2)ds, \tag{5.6}$$

where $K = \left[\|\varphi_1\|^2 + c_1\|\varphi_0\|_{\mathcal{V}}^2 + 2C_2k\|\varphi_0\|_{\mathcal{V}}^2 + 2C_3\right]$, and $L = lc_1 + 2lC_2k$.

Let $\tilde{\varepsilon} = \min(1, \varepsilon)$, $L_1 = \frac{L}{\tilde{\varepsilon}}$, $K_1 = \frac{K}{\tilde{\varepsilon}}$, $K_2 = \left(2lC_3 + \|f\|_{\mathcal{V}_2^*}^2(2\delta)^{-1}\right)\frac{1}{\tilde{\varepsilon}l}$ and $K_3 = \frac{2C_1}{\tilde{\varepsilon}}$. With these, using Gronwall's inequality, we get:

$$\|w_t(t)\|^2 + \|w(t)\|_{\mathcal{V}}^2 \leq K_1e^{-lt} + K_2(1-e^{-lt}) + K_3$$
$$+L_1\int_0^t e^{l(s-t)}\left[K_1e^{-ls} + K_2(1-e^{-ls}) + K_3\right]e^{\int_s^t L_1e^{l(\theta-t)}d\theta}ds$$
$$\leq K_1e^{-lt} + K_2(1-e^{-lt}) + K_3$$
$$+L_1\int_0^t \left(e^{-lt}(K_1-K_2) + (K_2+K_3)e^{l(s-t)}\right)e^{\frac{L_1}{l}}ds$$
$$\leq K_1e^{-lt} + K_3 + K_2(1-e^{-lt}) + L_1e^{\frac{L_1}{l}}|K_1-K_2|te^{-lt}$$
$$+L_1e^{\frac{L_1}{l}}\frac{K_2+K_3}{l}. \tag{5.7}$$

Here only $K_1 = \frac{1}{\tilde{\varepsilon}}\left(\|\varphi_1\|^2 + c_1\|\varphi_0\|_{\mathcal{V}}^2 + 2C_2k\|\varphi_0\|_{\mathcal{V}}^2 + 2C_3\right)$ depends on the initial conditions (L_1, K_2, K_3 are independent of them), so given any $\Phi = \begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix} \in \mathcal{V}\times\mathcal{H}$ and $\varepsilon > 0$ there exists a $t_0 > 0$ such that for $t > t_0$, we have $|e^{-lt}K_1| < \frac{\varepsilon}{2}$ and $|K_1 - K_2|te^{-lt} < \frac{\varepsilon}{2}$, so

$$\|w_t(t)\|^2 + \|w(t)\|_{\mathcal{V}}^2 \leq \varepsilon + K_3 + K_2 + L_1e^{\frac{L_1}{l}}(K_2+K_3)\,l^{-1}. \tag{5.8}$$

This means that the weak dynamical system is weak point-dissipative, i.e., a ball of radius

$$R = \sqrt{K_3 + K_2 + L_1e^{\frac{L_1}{l}}\frac{K_2+K_3}{l}}$$

in $\mathcal{V}\times\mathcal{H}$ attracts the trajectories starting from any element of $\mathcal{V}\times\mathcal{H}$. (Actually, (5.7) shows strong point dissipativeness, which implies weak point dissipativeness.) The estimate (5.7) also shows that the dynamical system is bounded. To justify that these estimates are true for the weak solution we have to consider the Galerkin approximations and their properties (see Lemma 5.1 [4]). **Q.E.D.**

6 Asymptotic Behavior of Weak Solutions

Our first observation is that the system (1.1)-(1.3) possesses a Lyapunov function, i.e. there exist a function $F : \mathcal{V}\times\mathcal{H} \to I\!\!R$ such that

$$F\left(S_t\left(\Phi\right)\right) \leq F\left(\Phi\right) \tag{6.1}$$

for every $\Phi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in \mathcal{V} \times \mathcal{H},\ t \geq 0$. Indeed, take

$$F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\sigma_1(u,u) + G(\mathcal{N}u) - \mathrm{Re}\langle f, u\rangle_{\mathcal{V}_2^*,\mathcal{V}_2}.$$

For the weak solution $\begin{pmatrix} w \\ w_t \end{pmatrix}$ of (1.1)-(1.3), (5.1) gives:

$$\frac{d}{dt}\left\{\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\sigma_1(w,w) + G(\mathcal{N}w)\right\} - \mathrm{Re}\langle f, w_t\rangle_{\mathcal{V}_2^*,\mathcal{V}_2}$$

$$\leq -\mathrm{Re}\,\sigma_2(w_t, w_t) \leq -k_2\|w_t\|^2_{\mathcal{V}_2}.$$

Integrating from 0 to t we get:

$$\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\sigma_1(w,w) + G(\mathcal{N}w) - \mathrm{Re}\langle f, w\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \leq$$

$$\frac{1}{2}\|w_t(0)\|^2 + \frac{1}{2}\sigma_1(w(0),w(0)) + G(\mathcal{N}w(0)) - \mathrm{Re}\langle f, w(0)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2}$$

$$-k_2\int_0^t \|w_\tau\|^2_{\mathcal{V}_2} d\tau,$$

which shows that F is decreasing, so it satisfies (6.1).

Let $\omega(\Phi)$ denote the weak ω-limit set of $\Phi \in \mathcal{V} \times \mathcal{H}$, i.e.

$$\omega(\Phi) = \cap_{s\geq 0}\left(\overline{\cup_{t\geq s} S_t(\Phi)}^{w}\right).$$

Since trajectories are bounded, we know that $\omega(\Phi)$ is invariant under S_t, i.e. $S_t(\omega(\Phi)) = \omega(\Phi)$ (see [9], Lemma 2.1). Since our Lyapunov function is bounded on compact subsets of $\mathcal{V} \times \mathcal{H}$ and the map $\Psi \to F(\Psi) - F(S_t(\Psi))$ is lower semicontinuous in $\mathcal{V} \times \mathcal{H}$, where $\mathcal{V} \times \mathcal{H}$ is equipped with the weak topology, it follows that if $\Psi \in \omega(\Phi)$ then

$$F(S_t(\Psi)) = F(\Psi) \text{ for all } t \geq 0, \tag{6.2}$$

i.e., in the ω-limit set the Lyapunov function is constant along trajectories (see [7], Proposition 3.1.). However, our Lyapunov function can satisfy (6.2) only if $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ is a stationary solution of (1.1). So we arrive at the following:

Theorem 6.1 *Let* $\begin{pmatrix} w \\ w_t \end{pmatrix}$ *be the weak solution of (1.1)-(1.3). Then for any sequence* $\{t_n\}$, $t_n \to \infty$, *there exists a subsequence again denoted by* $\{t_n\}$ *such that*

$$\begin{pmatrix} w(t_n) \\ w_t(t_n) \end{pmatrix} \to \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$

weakly in $\mathcal{V} \times \mathcal{H}$ *as* $t_n \to \infty$*, where* Ψ *is a stationary solution of (1.1).*

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