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Local Controllability of a Nonlinear Shallow Spherical Shell*

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Abstract

We consider the problem of local controllability from the origin for a coupled nonlinear shell via boundary controls.

1 Introduction and Problem Statement

We consider the problem of controlling a nonlinear model of a thin, shallow spherical shell from the origin to a point in some neighborhood of the origin by means of boundary controls. The motion of the shell in question is described by the following system of nonlinear equations in $Q \equiv (0, \rho_0) \times (0, \infty)$:

$$\left.\begin{array}{r} u_{tt} \ + \ \frac{e}{R} v_{tt} + b_1^2(\rho) u_t - L(u) + \frac{1+\gamma}{R} w' - \frac{e}{R} L(v) \\ -[v' + \frac{1-\gamma}{2\rho} v] v + \frac{vs}{R} = 0 \\ \\ w_{tt} - e(v' + \frac{v}{\rho})_{tt} + b_2^2(\rho) w_t + \frac{e}{\rho}[L(v)\rho]' \\ -\frac{1+\gamma}{\rho}(\frac{u}{R}\rho)' + \frac{2(1+\gamma)}{R^2} w - \frac{1+\gamma}{2R} v^2 - \frac{1}{\rho}(vs\rho)' = 0 \end{array}\right\}, \qquad (1.1)$$

where we have introduced the notation,

$$\left.\begin{array}{lll} v \equiv \frac{u}{R} + w'; & s \equiv N + \gamma N_0; & N \equiv u' - \frac{w}{R} + \frac{1}{2} v^2; \\ N_0 \equiv \frac{u}{\rho} - \frac{w}{R}; & & L(u) \equiv u'' + \frac{u'}{\rho} - \frac{u}{\rho^2} \end{array}\right\}. \qquad (1.2)$$

Here, R denotes the radius of the middle surface of the spherical cap, $\gamma \in (-1, \frac{1}{2})$ is Poisson's ratio, u and w represent meridional and radial

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displacements, respectively, with respect to arclength, $\rho = R\theta$. We assume the shallowness condition for this shell, $\theta \in [0, \theta_0]$, where θ_0 is sufficiently small so that $\sin\theta_0 \approx \theta_0$ and $\rho_0 = R\theta_0 \ll \frac{1}{2}$. Since this is a problem in one spatial dimension, we denote differentiation with respect to ρ by $'$. The functions $b_i \in L^\infty(0, \rho_0)$ represent a light damping in the system and are assumed to be positive (but not necessarily uniformly positive) on $(0, \rho_0)$. Moreover, we assume that these functions are "sufficiently small" in $L^\infty(0, \rho_0)$, as will be described more fully later. Also, $e = \frac{1}{2}h^2$, where h is the thickness of the shell.

To control from the origin to a point, we impose the initial conditions

$$u(0, \cdot\,) = 0,\ u_t(0, \cdot\,) = 0,\ w(0, \cdot\,) = 0,\ w_t(0, \cdot\,) = 0. \tag{1.3}$$

We will control the shell by means of boundary controls, implemented through moments and forces applied to the open edge of the shell, while keeping the shell clamped at the apex:

$$u = w = w' = 0, \quad at \quad \rho = 0 \quad for \quad t > 0 \tag{1.4}$$

and

$$\left\{ \begin{array}{rl} s & = a_1 \\ ev' & = a_2 \\ eL(v) - vs - ev_{tt} & = a_3 \end{array} \right. \qquad at\ \rho = \rho_0,\ t > 0. \tag{1.5}$$

In order to handle the nonconstant, singular coefficients in this model, we will need to adapt the traditional Sobolev spaces to incorporate these factors. We now recall the appropriate weighted Sobolev spaces, which were first introduced with regard to the linear problem associated with (1.1) in [8].

Function spaces

The first step is to introduce an appropriate replacement for the space $L^2(0, \rho_0)$. This space will be the pivot space for finding the duals of the other Hilbert spaces, soon to be introduced.

$$L^2_\rho(0, \rho_0) \text{ with norm } \|u\|^2_{L^2_\rho} \; = \; \left\{ \int_0^{\rho_0} u^2 \rho \; d\rho \right\}^{\frac{1}{2}}. \tag{1.6}$$

We now describe what will be the state space for our variable u:

$$\mathcal{U}^1_\rho(0, \rho_0) \; = \; \left\{ u \; : \; \frac{u}{\sqrt{\rho}}, u'\sqrt{\rho} \in L_2(0, \rho_0),\ u(0) = 0 \right\}, \tag{1.7}$$

with norm

$$\|u\|_{\mathcal{U}^1_\rho} \; = \; \left\{ \int_0^{\rho_0} \left[\frac{u^2}{\rho} \; + \; (u')^2 \rho \right] d\rho \right\}^{\frac{1}{2}}. \tag{1.8}$$

Note that since $\frac{u}{\sqrt{\rho}}$ and $u'\sqrt{\rho}$ are in $L^2(0,\rho_0)$, we have that $uu' \in L^1(0,\rho_0)$ and hence the function

$$u^2(\rho) = u^2(\rho_0) + \int_{\rho_0}^{\rho} \frac{d}{dr}(u^2)dr = u^2(\rho_0) + 2\int_{\rho_0}^{\rho} uu'dr \tag{1.9}$$

is absolutely continuous on $[0,\rho_0]$. In particular, this implies that $u(0)$ is well-defined for all $u \in \mathcal{U}_\rho^1$.

For the variable w, we define the space:

$$\mathcal{W}_\rho^2(0,\rho_0) \;=\; \{w \;:\; w\sqrt{\rho} \in L_2(0,\rho_0),\; w' \in \mathcal{U}_\rho^1(0,\rho_0)\} \tag{1.10}$$

with norm

$$\|w\|_{\mathcal{W}_\rho^2} \;=\; \left\{\int_0^{\rho_0} w^2\rho d\rho \;+\; \|w'\|^2_{\mathcal{U}_\rho^1}\right\}^{\frac{1}{2}}. \tag{1.11}$$

By the same argument as was used for u, we have that $w \in \mathcal{W}_\rho^2$ implies that w is absolutely continuous on $[0,\rho_0]$.

We define the product space which will be the state space for the velocity terms, $[u_t, w_t]$ by:

$$\begin{aligned}\mathcal{V}_\rho^1(0,\rho_0) \;=\; & \{(u,w) \in L_\rho^2 \times L_\rho^2 \;:\; v = \tfrac{u}{R} + w' \in L_\rho^2(0,\rho_0) \\ & or,\; equivalently,\; w' \in L_\rho^2(0,\rho_0)\}\end{aligned} \tag{1.12}$$

with norm

$$\|(u,w)\|_{\mathcal{V}_\rho^1} \;=\; \{\|u\|^2_{L^2_\rho} + \|w\|^2_{L^2_\rho} + e\|v\|^2_{L^2_\rho}\}^{\frac{1}{2}}. \tag{1.13}$$

For the space of the controls, we define $\mathcal{U} \equiv [L^2([0,T]; R_{\rho_0})]^3$, where $R^3_{\rho_0}$ is the usual R^3 with a weight of ρ_0:

$$([l_1,l_2,l_3],[k_1,k_2,k_3])_{R^3_{\rho_0}} \;=\; (l_1k_1 + l_2k_2 + l_3k_3)\rho_0. \tag{1.14}$$

Finally, we introduce the state space for the system, $\mathcal{E}$, which is defined as

$$\mathcal{E} = [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1. \tag{1.15}$$

Note : We will henceforth drop the specification $(0,\rho_0)$ in the notation of the above spaces.

2 Statement of the Main Result

We now state the main theorem of this paper.

Theorem 2.1 *The system (1.1) with $b_i(\rho)$ sufficiently small, is locally controllable from the origin on the space $\mathcal{E} \equiv \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \times \mathcal{V}_\rho^1$ within the class of controls $\mathbf{a} \in \mathcal{U}$ for $T > \widetilde{T}_0$, where $\widetilde{T}_0$ is the time for exact controllability for the linearized system corresponding to (1.1). That is to say, given homogeneous initial data,*

$$u(0,\rho) = w(0,\rho) = u_t(0,\rho) = w_t(0,\rho) = 0 \tag{2.1}$$

and $T > \widetilde{T}_0$, there exist boundary controls $\mathbf{a} \in \mathcal{U}$ such that the reachable set for the system (1.1) contains some ball of radius $R > 0$ in the space $\mathcal{E}$:

$$\{[u(T,\rho), w(T,\rho), u_t(T,\rho), w_t(T,\rho)] : \mathbf{a} \in \mathcal{U}\} \supset \mathcal{B}_R(\mathcal{E}). \tag{2.2}$$

Remark: We note that the system (1.1) is well-posed for the time reversed system, $t \longrightarrow -t$. Consequently, we have for $b_i^2(\rho) \longrightarrow -b_i^2(\rho)$ that this is equivalent to driving a sufficiently small initial state to the origin in time $T > \widetilde{T}_0$.

An outline for the proof is as follows. First, we develop the abstract formulation of the system (1.1) in terms of the underlying semigroup. We will then prove exact controllability for the linearized system corresponding to (1.1), extending the results in [14] to the present model. Following this, we prove that the nonlinear abstract model is well-posed on a space $\mathbf{E}$ which is norm equivalent to $\mathcal{E}$. We then will construct a control to state mapping and prove that this mapping is a homeomorphism of a neighborhood of the origin in the control space $\mathcal{U}$ onto a neighborhood of the origin in the state space $\mathbf{E}$.

3 Abstract Formulation of the System (1.1)

In order to prove the main result, we will use a technique developed first by Lee and Markus (see [10]) for systems of ordinary differential equations. This technique has since been adapted for infinite dimensional systems. (See, for example, [5, 12, 6]. We note that the results in [6] correspond to global controllability results. This work is also quite beneficial for the current problem, with modifications on the assumptions). To employ this technique, we first must put the system (1.1) into an abstract o.d.e. framework over the Hilbert spaces defined above. To do this, we will recall several operators, which have previously been developed in [8, 14] for the linear system and in [1] for the nonlinear system.

The operators A, M, B and P. Let the operators A, M and B be as in [8] and P as in [14]. We recall here some of the properties of these operators. By Section 3 in [8], we have that

(i) A is positive, self-adjoint on $L^2_\rho \times L^2_\rho$, continuous on $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \longrightarrow [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho]'$;

(ii) M is positive, self-adjoint on $L^2_\rho \times L^2_\rho$, and is an isomorphism $\mathcal{V}^1_\rho$ onto $[\mathcal{V}^1_\rho]'$, (duality w.r.t. $L^2_\rho \times L^2_\rho$);

(iii) $\mathbf{E} = D(A^{\frac{1}{2}}) \times D(M^{\frac{1}{2}})$ is norm equivalent to $\mathcal{E} = \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \times \mathcal{V}^1_\rho$;

(iv) $B^* \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -u(\rho_0) \\ -v(\rho_0) \\ w(\rho_0) \end{bmatrix} : D(A^{1/2}) = \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \longrightarrow \mathbf{R}^3_{\rho_0}$.

From [14], we have that P is a self-adjoint perturbation on $L^2_\rho \times L^2_\rho$, which satisfies

$$P : \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \longrightarrow L^2_\rho \times L^2_\rho \quad \text{is continuous.} \tag{3.1}$$

Note: Throughout this work, the notation $\|\cdot\|$ without subscript will denote the operator norm over the appropriate space(s) as indicated by the operator in question.

The nonlinear boundary operator f. We define the nonlinear operator $f : D(A^{1/2}) \longrightarrow \mathbf{R}^3_{\rho_0}$ as in [1]:

$$f(\underline{u}) = \begin{pmatrix} -\frac{1}{2}v^2(\rho_0) \\ 0 \\ 0 \end{pmatrix}, \quad \underline{u} = (u, w); \quad v = \frac{u}{R} + w'. \tag{3.2}$$

Then we have that

$$\left(Bf(\underline{u}), \quad \begin{bmatrix} \bar{u} \\ \bar{w} \end{bmatrix} \right)_{L^2_\rho \times L^2_\rho} = \frac{1}{2}\bar{u}(\rho_0)v^2(\rho_0)\rho_0. \tag{3.3}$$

Moreover, by [1](Lemmas 4.1 & 4.2), we have that f, as defined in (3.2) is continuous on $D(A^{1/2}) \longrightarrow \mathbf{R}^3_{\rho_0}$ and is Frechét differentiable with derivative,

$$Df(\underline{u})[\underline{h}] = \begin{pmatrix} [(\frac{u}{R} + w')(\frac{h_1}{R} + h_2')](\rho_0) \\ 0 \\ 0 \end{pmatrix}; \qquad \underline{h} \in D(A^{1/2}). \tag{3.4}$$

The nonlinear operator $\mathcal{F}$. We define the operator $\mathcal{F} : D(A^{1/2}) \longrightarrow [\mathcal{V}^1_\rho]'$ as in [1]:

$$(\mathcal{F}\underline{u}, \underline{g}) = -(v'v, g_1)_{L^2_\rho} - \frac{1-\gamma}{2}(v^2, \frac{g_1}{\rho})_{L^2_\rho} + \frac{1}{R}(vs, g_1)_{L^2_\rho}$$

$$
\begin{aligned}
& - \frac{1+\gamma}{2R}(v^2, g_2)_{L^2_\rho} \; + \; (vs, g_2')_{L^2_\rho} \\
= \; & -(v'v, g_1)_{L^2_\rho} \; - \; \frac{1-\gamma}{2}(v^2, \frac{g_1}{\rho})_{L^2_\rho} \; - \; \frac{1+\gamma}{2R}(v^2, g_2)_{L^2_\rho} \\
& + (vs, v_g)_{L^2_\rho}, \qquad (3.5)
\end{aligned}
$$

where $(\cdot,\cdot)$ represents the duality pairing between $\mathcal{V}^1_\rho$ and $[\mathcal{V}^1_\rho]'$, $\underline{\mathrm{u}} = (u, w)$, $\underline{\mathrm{g}} = (g_1, g_2)$ and $v_g = \frac{g_1}{R} + g_2'$.

By Lemmas 5.1 & 5.2 in [1], we have that $\mathcal{F} : D(A^{1/2}) \longrightarrow [\mathcal{V}^1_\rho]'$ is continuous and continuously Frechét differentiable, with derivative given by

$$
\begin{aligned}
(D\mathcal{F}(\underline{\mathrm{u}})[\underline{\mathrm{h}}], \underline{\mathrm{g}}) \; = & -([v'v_h \; + \; vv_h'], g_1)_{L^2_\rho} \; - \; (1-\gamma)(vv_h, \tfrac{g_1}{\rho})_{L^2_\rho} \\
& - \tfrac{(1+\gamma)}{R}(vv_h, g_2)_{L^2_\rho} \; + \; (v[h_1' \; + \; \gamma\tfrac{h_1}{\rho} \; - \; \tfrac{1+\gamma}{R}h_2], v_g)_{L^2_\rho} \\
& + (v_h(v^2 \; + \; s), v_g)_{L^2_\rho} \qquad (3.6)
\end{aligned}
$$

for $\underline{\mathrm{g}} \in \mathcal{V}^1_\rho$ and with $v_h = \frac{h_1}{R} + h_2'$, where $\underline{\mathrm{h}} \equiv (h_1, h_2) \in \mathcal{V}^1_\rho$.

Remark: We note that both $Df(\underline{\mathrm{u}})$ and $D\mathcal{F}(\underline{\mathrm{u}})$ are continuous near the origin, with $Df(0) = 0$ in $\mathbf{R}^3_{\rho_0}$ and $D\mathcal{F}(0) = 0$ in $[\mathcal{V}^1_\rho]'$.

We now introduce the perturbation operator, $P_2 : \mathcal{V}^1_\rho \longrightarrow L^2_\rho \times L^2_\rho$, which is bounded and self-adjoint over $L^2_\rho \times L^2_\rho$:

$$
P_2 \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} b_1^2(\rho) & 0 \\ 0 & b_2^2(\rho) \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}. \qquad (3.7)
$$

Combining the above definitions proves

Proposition 3.1 *The abstract second order model of the system (1.1) with* $\underline{u} = (u, w)$ *and* $\mathbf{a} \in \mathcal{U}$ *is given by*

$$
M\underline{u}_{tt} \; + \; (A+P)\underline{u} \; + \; P_2\underline{u}_t \; + \; B(\mathbf{a}) \; + \; B(f(\underline{u})) \; = \; -\mathcal{F}(\underline{u}) \qquad (3.8)
$$

where the operators A, B, M, *and* P *are as in [8] and* f, $\mathcal{F}$ *and* P_2 *are defined in (3.2), (3.5) and (3.7), respectively.*

The first-order model corresponding to (3.8) is

$$
\begin{aligned}
\tfrac{d}{dt}\begin{pmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{u}}_t \end{pmatrix} \; - \; \begin{bmatrix} 0 & I \\ -M^{-1}(A+P) & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{u}}_t \end{pmatrix} \; - \; \mathcal{P}_2 \begin{pmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{u}}_t \end{pmatrix} \\
- \; \mathcal{B}(\mathbf{a}) \; - \; \mathcal{B}(f(\underline{\mathrm{u}})) = \mathcal{F}_i \begin{pmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{u}}_t \end{pmatrix}, \qquad (3.9)
\end{aligned}
$$

or equivalently,

$$\frac{d}{dt}(\mathbf{u}) \equiv \mathcal{A}(\mathbf{u}) + \mathcal{P}_2(\mathbf{u}) + \mathcal{B}(\mathbf{a}) + \ \mathcal{B}(f(\underline{\mathrm{u}})) + \mathcal{F}_i(\mathbf{u}), \tag{3.10}$$

where $\mathbf{u} = (\underline{\mathrm{u}}, \underline{\mathrm{u}}_t) \in \mathbf{E}$. Here, for convenience, we have introduced the boundary operator $\mathcal{B} : \mathcal{U} \longrightarrow [\mathbf{E}]'$, the interior nonlinear operator $\mathcal{F}_i$: $\mathbf{E} \longrightarrow \mathbf{E}$ and the perturbation operator $\mathcal{P}_2 : \mathbf{E} \longrightarrow \mathbf{E}$, which are defined as:

$$\mathcal{B}(\cdot) = \begin{pmatrix} 0 \\ -M^{-1}B(\cdot) \end{pmatrix}, \qquad \mathcal{P}_2(\mathbf{u}) = \begin{pmatrix} 0 \\ -M^{-1}P_2(\underline{\mathrm{u}}_t) \end{pmatrix}$$
$$\text{and} \qquad \mathcal{F}_i(\mathbf{u}) = \begin{pmatrix} 0 \\ -M^{-1}\mathcal{F}(\underline{\mathrm{u}}) \end{pmatrix}. \tag{3.11}$$

4 Proof of Main Result

Exact controllability of linearized system
It was proven in [14] that the linearized version of system (3.10) with the operator $\mathcal{P}_2 \equiv 0$ (i.e. $\mathcal{P}_2 = f = \mathcal{F} \equiv 0$), is exactly controllable on the space $\mathbf{E}$. We extend this result to the linearized system for (3.10) with $\mathcal{P}_2 \neq 0$.

Lemma 4.1 *Let $b_1^2(\rho)$ and $b_2^2(\rho)$ sufficiently small in $L^\infty(0, \rho_0)$. Then the linear system (3.10) with $f = \mathcal{F} = 0$, given by*

$$\mathbf{u}_t(t) = \mathcal{A}\mathbf{u}(t) + \mathcal{P}_2\mathbf{u}(t) + \mathcal{B}(\mathbf{a}), \qquad \mathbf{u}(0) \equiv \mathbf{u}_0 = 0, \tag{4.1}$$

is exactly controllable on the space $\mathbf{E}$ in time $T > \widetilde{T}_0$ sufficiently large. This $\widetilde{T}_0$ is related to the constants $T_0,\ C$ of equation (4.3) in [14] by

$$\widetilde{T}_0 = T_0 + \frac{1}{C}\|\mathcal{P}_2\|. \tag{4.2}$$

Proof: Since $\mathcal{P}_2 : C([0,T]; \mathbf{E}) \longrightarrow C([0,T]; \mathbf{E})$ is a self-adjoint, lower order perturbation of the operator $\mathcal{A}$, the "controllability inequality" (see (4.3) in [14]) becomes

$$\int_0^T \|\mathcal{B}^* e^{\mathcal{A}^*(T-t)}\mathbf{h}\|_{\mathcal{U}} dt + \|\mathcal{P}_2^*\mathbf{h}\|_{\mathbf{E}} \geq C_T\|\mathbf{h}\|_{\mathbf{E}} \qquad \text{for all } \mathbf{h} \in \mathbf{E}, \tag{4.3}$$

so that

$$\begin{aligned} \int_0^T \|\mathcal{B}^* e^{\mathcal{A}^*(T-t)}\mathbf{h}\|_{\mathcal{U}} dt \quad &\geq \quad (C_T - \|\mathcal{P}_2^*\|)\|\mathbf{h}\|_{\mathbf{E}} \\ &\equiv \quad \widetilde{C}_T\|\mathbf{h}\|_{\mathbf{E}}, \end{aligned} \tag{4.4}$$

where $C_T = C(T - T_0) > 0$ and $\widetilde{C}_T = C(T - T_0 - \frac{1}{C}\|\mathcal{P}_2^*\|) > 0$ as long as

$$T > T_0 + \frac{1}{C}\|\mathcal{P}_2^*\| = T_0 + \frac{1}{C}\|\mathcal{P}_2\|. \tag{4.5}$$

But this holds whenever $(\|b_1^2(\rho)\|_{L^\infty(0,\rho_0)} + \|b_2^2(\rho)\|_{L^\infty(0,\rho_0)})$ is sufficiently small. In particular, (4.4) implies that the perturbed system, (4.1) is exactly controllable using the same controls as for the unperturbed system (i.e. with $\mathcal{P}_2 \equiv 0$).

Remark: Since $\mathcal{P}_2$ is a lower order perturbation of the operator $\mathcal{A}$, we may include it as part of the generator of the semigroup for the linearized system:

$$\widetilde{\mathcal{A}} = \mathcal{A} + \mathcal{P}_2. \tag{4.6}$$

For use in the proof of Theorem 2.1, we introduce four operators which will simplify the writing of the solution to (3.10). Similar operators are developed in the arguments found in [6] for a global controllability result concerning semilinear waves.

Let $(\mathcal{L}\mathbf{a})(t) : \mathcal{U} \longrightarrow C([0,T];\mathbf{E})$ be given by

$$(\mathcal{L}\mathbf{a})(t) = \int_0^t e^{\widetilde{\mathcal{A}}(t-\tau)}\mathcal{B}(\mathbf{a}(\tau))d\tau \tag{4.7}$$

and $\mathcal{L}_T\mathbf{a} = (\mathcal{L}\mathbf{a})(T)$. Also, let $(\mathcal{R}\mathbf{u})(t) : C([0,T];\mathbf{E}) \longrightarrow C([0,T];\mathbf{E})$ be given by

$$(\mathcal{R}\mathbf{u})(t) = \int_0^t e^{\widetilde{\mathcal{A}}(t-\tau)}\mathbf{u}(\tau)d\tau \tag{4.8}$$

with $\mathcal{R}_T\mathbf{u} = (\mathcal{R}\mathbf{u})(T)$. With these definitions and Lemma 4.1, we have the continuity of the operators $\mathcal{L}$, $\mathcal{L}_T$, $\mathcal{R}$, $\mathcal{R}_T$, and in particular,

$$\begin{aligned} &\mathcal{L}_T : \mathcal{U} \longrightarrow \mathbf{E} \text{ is continuous} \\ \text{and} \quad & \\ &\mathcal{R}_T : C([0,T];\mathbf{E}) \longrightarrow \mathbf{E} \text{ is continuous.} \end{aligned} \tag{4.9}$$

Lemma 4.2 Well-posedness. *Let*

$$[u(0,\rho), w(0,\rho), u_t(0,\rho), w_t(0,\rho)] \equiv \mathbf{u}_0 \in \mathbf{E}$$

be given and let $\mathbf{a} \in \mathcal{U}$. *Then the system (1.1) admits a unique weak solution* $\mathbf{u}(t) \equiv [u(t,\rho), w(t,\rho), u_t(t,\rho), w_t(t,\rho)] \in C([0,T];\mathbf{E})$.

Proof: By the results of [1], we have a solution of the form:

$$\mathbf{u}(t) = e^{\widetilde{\mathcal{A}}}\mathbf{u}_0 + \mathcal{L}(\mathbf{a} + f(\mathbf{u}))(t) + \mathcal{R}\mathcal{F}_i(\mathbf{u})(t), \tag{4.10}$$

where $\mathcal{L}$ and $\mathcal{R}$ are as in (4.7) and (4.8), respectively.

Construction of the control to state map. By Lemma 4.1, we know that given any desired terminal state, $\mathbf{u}(T) = \mathbf{u}_T$, for the system (4.1) there exists a control $\mathbf{a} \in \mathcal{U}$ so that

$$\mathcal{L}_T\mathbf{a} \equiv (\mathcal{L}\mathbf{a})(T) = \mathbf{u}_T. \tag{4.11}$$

We note that by its very nature, $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset \mathcal{U}$, which denotes the orthogonal compliment of the null space of $\mathcal{L}_T$ in $\mathcal{U}$:

$$\mathcal{U} = \mathcal{N}(\mathcal{L}_T) \oplus [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{4.12}$$

Here, $\oplus$ denotes the orthogonal decomposition of $\mathcal{U}$. We will follow the notation as in [6], defining the pseudoinverse,

$$\mathcal{L}_T^{\#} = (\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^{-1} : \mathbf{E} \longrightarrow [\mathcal{N}(\mathcal{L}_T)]^\perp, \tag{4.13}$$

where

$$\begin{cases} \mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp} = \mathcal{L}_T \text{ restricted to } [\mathcal{N}(\mathcal{L}_T)]^\perp, \\ \mathcal{L}_T\mathcal{L}_T^{\#} = \text{ Identity on } \mathbf{E}, \\ \mathcal{L}_T^{\#}\mathcal{L}_T = \mathbf{\Pi_T} = \text{ orthogonal projection of } \mathbf{E} \text{ onto } [\mathcal{N}(\mathcal{L}_T)]^\perp, \\ \mathcal{L}_T^{\#}\mathcal{L}_T = \text{ Identity on } [\mathcal{N}(\mathcal{L}_T)]^\perp. \end{cases} \tag{4.14}$$

We will henceforth restrict our attention to controls $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, redefining our problem to consider only this subset of the controls. We note that once such a control is found for our system, any control $\tilde{\mathbf{a}} \in \mathcal{U}$ which produces the desired terminal state may be written in the form $\mathbf{\Pi_T}\tilde{\mathbf{a}} = \mathbf{a}$ with $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$.

We now return to the solution to the nonlinear system, (3.10), which is guaranteed to exist by Lemma 4.2:

$$\mathbf{u}(t) = \mathcal{L}\mathbf{a}(t) + \mathcal{L}f(\mathbf{u}(\mathbf{a}))(t) + \mathcal{R}\mathcal{F}_i(\mathbf{u}(\mathbf{a}))(t); \quad \mathbf{u}(0) \equiv \mathbf{u}_0 = 0, \tag{4.15}$$

where $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Defining the control to state map $\mathcal{M}_T : \mathcal{U} \longrightarrow \mathbf{E}$ by

$$\mathcal{M}_T(\mathbf{a}) = \mathcal{L}_T\mathbf{a} + \mathcal{L}_T f(\mathbf{u}(\mathbf{a})) + \mathcal{R}_T\mathcal{F}_i(\mathbf{u}(\mathbf{a})), \tag{4.16}$$

we wish to show that there exist constants, r, $R > 0$ such that $\mathcal{M}_T : \mathcal{B}_r([\mathcal{N}(\mathcal{L}_T)]^\perp) \longrightarrow \mathcal{B}_R(\mathbf{E})$ is a homeomorphism. To do this, we will employ the implicit function theorem for Banach spaces (see [2]).

We first show that $\mathcal{M}_T(0) = 0$. Since $\mathbf{u}_0 = 0$ and $\mathbf{a} = 0$ together imply that $\mathbf{u}(t) \equiv 0$, and since $\mathcal{F}_i(0) = 0$ and $f(0) = 0$, we have $\mathcal{M}_T(0) = 0$.

Differentiability of the Map $\mathcal{M}_T$.

To prove that $\mathcal{M}_T$ is Frechét differentiable in a neighborhood of the origin, we will need the following lemma.

Lemma 4.3 *The solution* $\mathbf{u}(t) = \mathbf{u}(\mathbf{a}, t)$ *is differentiable with respect to the control* $\mathbf{a}$ *in a neighborhood of the origin, with derivative,*

$$\frac{d\mathbf{u}}{d\mathbf{a}} = [I - \mathcal{L}Df(\mathbf{u}) - \mathcal{R}D\mathcal{F}_i(\mathbf{u})]^{-1}\mathcal{L}. \tag{4.17}$$

Proof: We begin by formally differentiating the equation (4.15) with respect to the control $\mathbf{a}$:

$$\frac{d\mathbf{u}}{d\mathbf{a}} = \mathcal{L} + \mathcal{L}Df(\mathbf{u}(\mathbf{a}))\frac{d\mathbf{u}}{d\mathbf{a}} + \mathcal{R}D\mathcal{F}_i(\mathbf{u}(\mathbf{a}))\frac{d\mathbf{u}}{d\mathbf{a}}, \tag{4.18}$$

or equivalently,

$$[I - \mathcal{L}Df - \mathcal{R}D\mathcal{F}_i]\frac{d\mathbf{u}}{d\mathbf{a}} = \mathcal{L}. \tag{4.19}$$

Claim: The operator, $I - \mathcal{L}Df(\mathbf{u}) - \mathcal{R}D\mathcal{F}_i(\mathbf{u})$, is boundedly invertible on a ball of radius $\widetilde{R}$ in $\mathbf{E}$, $\mathcal{B}_{\widetilde{R}}(\mathbf{E})$, for some $\widetilde{R} > 0$, sufficiently small.

Proof of Claim: Since $Df(\mathbf{u})$ and $D\mathcal{F}_i(\mathbf{u})$ are both continuously Frechét differentiable in a neighborhood of the origin, with $Df(0) = 0$ in $\mathcal{U}$ and $D\mathcal{F}_i(0) = 0$ in $\mathbf{E}$, we have that, for $\mathbf{h} \in \mathbf{E}$,

$$\|D\mathcal{F}_i(\mathbf{u})[\mathbf{h}]\|_{C([0,T];E)} \le C_1(\|\mathbf{u}\|_{C([0,T];E)})\|\mathbf{h}\|_{C([0,T];E)} \tag{4.20}$$

and

$$\|Df(\mathbf{u})[\mathbf{h}]\|_{\mathcal{U}} \le C_2(\|\mathbf{u}\|_{C([0,T];E)})\|\mathbf{h}\|_{C([0,T];E)}, \tag{4.21}$$

where $C_i(\|\mathbf{u}\|_{C([0,T];E)}) \longrightarrow 0$ as $\|\mathbf{u}\|_{C([0,T];E)} \longrightarrow 0$. In fact, it follows from the results in [1],that

$$\begin{aligned} C_1(\|\mathbf{u}\|_{C([0,T];E)}) &= C \cdot (\|\mathbf{u}\|_{C([0,T];E)} + \|\mathbf{u}\|^2_{C([0,T];E)}), \\ C_2(\|\mathbf{u}\|_{C([0,T];E)}) &= C \cdot \|\mathbf{u}\|_{C([0,T];E)}. \end{aligned} \tag{4.22}$$

Consequently, we may take $\|\mathbf{u}\|_{C([0,T];E)} < \widetilde{R}$, sufficiently small so that

$$1 - C\widetilde{R}\|\mathcal{L}\| - C(\widetilde{R} + \widetilde{R}^2)\|\mathcal{R}\| > 0. \tag{4.23}$$

Then, for $\mathbf{u} \in C([0,T]; \mathcal{B}_{\widetilde{R}}(\mathbf{E}))$ we have the invertibility of the operator $[I - \mathcal{L}Df(\mathbf{u}) - \mathcal{R}D\mathcal{F}_i(\mathbf{u})]$, as desired.

We can see now that the derivative, $\frac{d\mathbf{u}}{d\mathbf{a}}$ as given by (4.18) is well-defined for $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$ and may be written as in (4.17), as desired.

Lemma 4.4 *The control to state map, $\mathcal{M}_T$ as defined in (4.16) is Frechét differentiable in a neighborhood of the origin in $[\mathcal{N}(\mathcal{L}_T)]^\perp$, with derivative $D\mathcal{M}_T(\mathbf{a})$ given by*

$$\begin{aligned} D\mathcal{M}_T(\mathbf{a})[\mu] &= \mathcal{L}_T[\mu] \\ &+ \{\mathcal{L}_T Df(\mathbf{u}) + \mathcal{R}_T D\mathcal{F}_i(\mathbf{u})\} [I - \mathcal{L}Df(\mathbf{u}) - \mathcal{R}D\mathcal{F}_i(\mathbf{u})]^{-1}\mathcal{L}[\mu]. \end{aligned} \tag{4.24}$$

Moreover, $D\mathcal{M}_T(0)$ is well-defined and boundedly invertible for $\mu \in [\mathcal{N}(\mathcal{L}_T)]^\perp$.

Proof: Differentiating equation (4.16) with respect to $\mathbf{a}$, substituting (4.17) in for $\frac{d\mathbf{u}}{d\mathbf{a}}$ gives us (4.4), which is a well-defined and bounded operator for $\mu \in \mathcal{U}$. Then, for $\mu \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, we have

$$D\mathcal{M}_T(0)[\mu] = \mathcal{L}_T\mu. \tag{4.25}$$

Since $\mathcal{L}_T$ is invertible on $[\mathcal{N}(\mathcal{L}_T)]^\perp$, with inverse $\mathcal{L}_T^\#$, as defined in (4.13), we have the lemma.

Completion of the proof of Theorem 2.1

By an application of the implicit function theorem, we have that $\mathcal{M}_T : \mathcal{B}_r([\mathcal{N}(\mathcal{L}_T)]^\perp) \longrightarrow \mathcal{B}_R(\mathbf{E})$ is an homeomorphism for some r, $R > 0$. This guarantees the existence of a solution to the equation

$$\mathcal{M}_T(\mathbf{a}) = \mathbf{u}_T \tag{4.26}$$

for any $\mathbf{u}_T \in \mathcal{B}_R(\mathbf{E})$ via control $\mathbf{a} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Consequently, we have the local controllability of system (3.10), or equivalently, for system (1.1), as desired.

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