Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 2, 1998, pp. 1–?

A Computational Study of the Representation Problem for Flow Control*

Diana Rubio†

Abstract

The problem of existence of functional gains for boundary control of parabolic problems remains incomplete. In particular, it is not yet known when LQR feedback operators have integral representations and even in the case where an integral representation exists there is no theory concerning the regularity of the kernels. There are cases where the feedback operator is known to be Hilbert-Schmidt and hence one has an integral representation with L_2 kernels. These kernels are called functional gains. Functional gains can be used to address sensor/actuator location problems and to design low dimensional compensators [2]. In this note we investigate this question for a flow control problem by using finite element numerical methods.

Key words: incompressible viscous flow, Boussinesq equations, analytic semigroups, feedback control, finite element method

AMS Subject Classifications: 35K55, 76D05, 20M20, 65N30, 93B52, 93B40

1 Problem Description

We consider a boundary control problem for a two dimensional thermal convection loop. A thermal convection loop consists of a viscous fluid contained in a circular pipe standing in a vertical plane. When the walls of the pipe are heated from below (creating a thermal difference between the top and the bottom of the pipe) a temperature gradient is caused by thermal expansion and the fluid tends to flow. On the other hand, the viscosity

*Received November 5, 1996; received in final form June 26, 1997. Summary appeared in Volume 8, Number 2, 1998. This paper was presented at the Conference on Computation and Control V, Bozeman, Montana, August 1996. The paper was accepted for publication by special editors John Lund and Kenneth Bowers.

†This research was supported in part by the Air Force Office of Scientific Research under grants F49620-93-1-0280 and F49620-96-1-0329.

and the thermal diffusivity resists these motions. When dissipative terms are overcome by the buoyancy force, fluid motion is created. This effect is called *buoyancy driven free convection.*

A simplified version of this problem is obtained when the Boussinesq model is considered. The Boussinesq approximation assumes that all fluid properties are constant, except for the density in the buoyancy term. The body force per unit mass, $\vec{F}$, is due only to gravitational acceleration and the buoyancy force per unit mass. This force is given by

$$\vec{F} = \vec{g} + \beta\ (T(t,r,\varphi) - T_0)(-\vec{g}),$$

where $\vec{g}$ is the gravity acceleration, β is the thermal expansion coefficient and T_0 is the bulk fluid temperature.

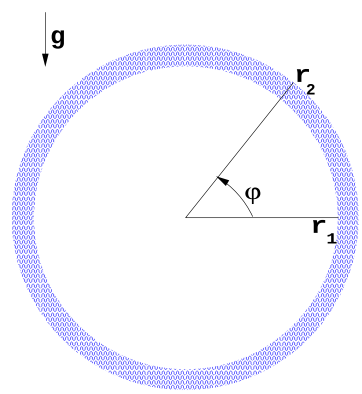

Figure 1.1: Description of the thermal convection loop

Figure 1.1 illustrates the thermal convection loop. The inner radius of the pipe is denoted by r_1 and the outer radius by r_2. The variable φ measures the angle counterclockwise from the horizontal position. We restrict ourselves to the case of a thin pipe where the width of the pipe is small as compared with the interior radius, i.e., $r_2 - r_1 << r_1$. In this case, the flow may be considered as a Poiseuille flow in a straight pipe of width $r_2 - r_1$, so that the velocity depends only on the radial coordinate. Therefore, we have

$$\vec{v}(t,r,\varphi) = v(t,r)\hat{\varphi} \tag{1.1}$$

where $\hat{\varphi}$ is a unit vector along the pipe.

The dynamics of an incompressible viscous fluid is modeled by the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = \rho\vec{F} - \frac{1}{\rho}\ \nabla p + \frac{\mu}{\rho}\ \nabla^2\vec{v} \tag{1.2}$$

where p, ρ and μ are the fluid's pressure, density and viscosity, respectively.

We assume zero velocity at the walls and that the fluid's temperature equals the wall's temperature. Thus, we have a Dirichlet boundary control problem. Following [4],[5], we integrate the Navier-Stokes equation along a circular path at a fixed radius r. It can be shown that after integrating, the term $(\vec{v}.\nabla)\vec{v}$ and the pressure term are eliminated.

The Laplacian operator is given in polar coordinates by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2 \varphi},$$

and we let $\nabla_r^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$. The dynamics of the fluid flow may be described by the following quasilinear infinite-dimensional distributed parameter system (see [7], [10]),

$$\frac{\partial v}{\partial t}(t,r) = \nu \nabla_r^2 v(t,r) + \frac{g\beta}{2\pi}\int_0^{2\pi} T(t,r,\varphi)\ \cos\varphi\ d\varphi, \tag{1.3}$$

$$\frac{\partial T}{\partial t}(t,r,\varphi) = -\frac{v(t,r)}{r}\ \frac{\partial T}{\partial \varphi}(t,r,\varphi) + \chi\ \nabla^2 T(t,r,\varphi), \tag{1.4}$$

with boundary conditions

$$v(t,r_1) = v(t,r_2) = 0,$$

$$T(t,r_1,\varphi) = T_1(t,\varphi) + w_1(t,\varphi), \qquad T(t,r_2,\varphi) = T_2(t,\varphi) + w_2(t,\varphi).$$

Here v denotes the fluid's velocity, T the fluid's temperature, ν the kinematic viscosity of the fluid, g the gravity acceleration constant, β the thermal expansion and χ the coefficient of thermal diffusivity. Note that the system is invariant under translation of T, thus the state $T(t,r,\varphi)$ may be interpreted as a difference in temperature from the bulk temperature. Here we take a bulk temperature of $60°F$.

The system (1.3)-(1.4) can be written in abstract form as follows

$$\dot{z}(t) = Az(t) + f(z(t)) + Bu(t), \quad t > 0, \qquad z(0) = z_0 \tag{1.5}$$

on the state space $H = L_2(\Omega_1) \times L_2(\Omega)$, where $\Omega_1 = [r_1, r_2]$, $\Omega_2 = [0, 2\pi)$, $\Omega = \Omega_1 \times \Omega_2$ and $\Gamma = \{r_1, r_2\} \times \Omega_2$. Here, $z(t) = (v(t,.), T(t,.,.))^T$ is the state. The linear operator A is defined on

$$\mathcal{D}(A) = \mathcal{D}(A_0) = [H^2(\Omega_1) \cap H_0^1(\Omega_1)] \times [H^2(\Omega) \cap H_0^1(\Omega)] \tag{1.6}$$

by $A = A_0 + A_1$ where

$$A_0 = \begin{pmatrix} \frac{\mu}{\rho}\ \nabla_r^2 & 0 \\ 0 & \chi\ \nabla^2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & \mathcal{I} \\ 0 & 0 \end{pmatrix}, \tag{1.7}$$

where $\mathcal{I} : L_2(\Omega) \to L_2(\Omega_1)$ is the bounded linear operator

$$[\mathcal{I}\omega](r) = \frac{g\beta}{2\pi} \int_0^{2\pi} \cos\varphi \; \omega(t,r,\varphi) d\varphi. \tag{1.8}$$

The nonlinear operator $f : H_0^1(\Omega_1) \times H_0^1(\Omega) \to H$ is defined by

$$[f(v(.), T(.,.))](r,\varphi) = \left(0, -\frac{v(r)}{r}\frac{\partial T}{\partial \varphi}(r,\varphi)\right)^T. \tag{1.9}$$

The control space is $U = L_2(\Gamma)$ and the input operator B is the unbounded linear operator

$$B = -\hat{A}\hat{D}. \tag{1.10}$$

Here $\hat{A}$ is the lifting of A from $\mathcal{D}(A)$ to H and $\hat{D} : L_2(\Gamma) \to H$ is given by $\hat{D}u = (0, Du)^T$. The operator $D : L_2(\Gamma) \to L_2(\Omega)$ is the Dirichlet map for the Laplacian ∇^2 on Ω, i.e., D is the bounded linear operator satisfying

$$Du = \omega \qquad \text{if and only if} \qquad \nabla^2\omega = 0 \qquad \text{and} \qquad \omega|_\Gamma = u. \tag{1.11}$$

It is well known that the Laplacian operator ∇^2 is dissipative and has only point spectrum $\sigma(\nabla^2)$ on the real axis. This observation leads to the following result.

Lemma 1 *The operator A_0 generates an analytic semigroup of contractions $S_0(t)$ defined on H. Moreover, there is a constant $\mu_0 > 0$ such that*

$$\|S_0(t)\| \leq e^{-\mu_0 t}. \tag{1.12}$$

Theorem 1 *The operator A defined in (1.6)-(1.8) generates an analytic semigroup $S(t)$ on H. Moreover, there are constants $M, \gamma > 0$ such that*

$$\|S(t)\| \leq Me^{-\gamma t}. \tag{1.13}$$

Proof: Observe that A is a bounded (compact) perturbation of A_0, thus A generates an analytic semigroup $S(t)$ on H (see [6], p.81). Let $\rho(A_0)$ be the resolvent set of the operator A_0 and $\rho(A)$ be the resolvent set of A. It is easy to see that $\rho(A_0) \subset \rho(A)$ and $\sigma(A) \subset \sigma(A_0)$. Thus,

$$\mu = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\} \leq \sup\{\text{Re}\lambda : \lambda \in \sigma(A_0)\} = \mu_0 < 0.$$

Consequently, (1.13) follows from Theorem 4.3, Chapter 4, in [6]. □

Since A_0 generates an analytic semigroup with $0 \in \rho(A)$, $(-A_0)^{1/2}$ is well defined (see [6], p. 69). Let V be the Hilbert space $H_0^1(\Omega_1) \times H_0^1(\Omega)$ equipped with the norm $\|z\|_V = \|(-A_0)^{1/2}z\|_H$. We have the following lemma.

Lemma 2 *Consider the non-linear operator $f : V \to H$ defined by*

$$f(z) = \begin{bmatrix} 0 \\ -\frac{v}{r}\frac{\partial T}{\partial \varphi}(.,.) \end{bmatrix}. \tag{1.14}$$

For any $z \in V$ there is a neighborhood $\mathcal{N}$ and a constant C such that

$$\|f(u) - f(w)\| \leq C\|u - w\|_V \tag{1.15}$$

for all $u, w \in \mathcal{N}$.

Proof: If $u, w \in V$, then f is well-defined and we have

$$\begin{aligned} \|f(u) - f(w)\|_H &= \left\| u_1 \frac{\partial u_2}{\partial \varphi} - w_1 \frac{\partial w_2}{\partial \varphi} \right\|_H \\ &\leq \left\| \frac{\partial u_2}{\partial \varphi} \right\|_H \|u_1 - w_1\|_H + \|w_1\|_H \left\| \frac{\partial u_2}{\partial \varphi} - \frac{\partial w_2}{\partial \varphi} \right\|_H \\ &\leq \|u\|_V \|u - w\|_V + \|w\|_V \|u - w\|_V. \end{aligned} \tag{1.16}$$

Let $z_0 \in V$ and assume that $\mathcal{N}(z_0, \delta)$ is a neighborhood of z_0. There is a constant C_1 such that if $z \in \mathcal{N}(z_0, \delta)$, then $\|z\|_V \leq C_1$. Hence, if $u,\ w \in \mathcal{N}(z_0, \delta)$, then

$$\|f(u) - f(w)\|_H \leq 2C_1 \|u - w\|_V,$$

and (1.15) holds for $C = 2C_1$. $\square$

Theorem 2 *If $u = 0$ and z_0 is sufficiently smooth, then there exists a $T > 0$ such that (1.5) has a unique local solution $z(t)$ on $[0, T)$.*

Proof: We consider the uncontrolled system

$$\begin{aligned} \dot{z} &= Az + f(z), \qquad t > 0, \\ z(0) &= z_0. \end{aligned}$$

We have shown that A generates an analytic semigroup with $0 \in \rho(A)$ and f satisfies (1.15). Then, the existence and uniqueness of a local solution to (1.5) with $u = 0$ follows from Theorem 3.1, Chapter 6, in Pazy [6]. $\square$

2 LQR Feedback Control

We linearize the system at the equilibrium point $v = 0, T = 0$. Since $f(0,0) = (0,0)^T$, the linearized system becomes

$$\dot{z}(t) = Az(t) + Bu(t), \quad t > 0, \qquad z(0) = z_0, \tag{2.17}$$

where A and B are the operators defined on 1.6-1.8 and 1.10, respectively.

The LQR problem is to minimize the quadratic cost defined by

$$J(u) = \int_0^\infty [\langle Qz(t), z(t)\rangle_H + \langle Ru(t), u(t)\rangle_U]\, dt \tag{2.18}$$

over all controls $u \in L_2([0,\infty);U)$, subject to the linear system (2.17). The state weighting operator for the LQR problem is

$$Q = \begin{pmatrix} Q_v & 0 \\ 0 & Q_T \end{pmatrix}$$

with $Q_v = q_v I_{L_2(\Omega_1)}, Q_T = q_T I_{L_2(\Omega)}$ and q_v, q_T positive constants. The operators $I_{L_2(\Omega_1)}, I_{L_2(\Omega)}$ denote the identity operators in $L_2(\Omega_1)$ and $L_2(\Omega)$, respectively. The control weighting operator is given by $R = q_u I_U$, where I_U denotes the identity operator on U and q_u is a positive constant.

Since (2.17) is a stable system, an optimal control exists and it is given in feedback form

$$u_{opt}(t) = -R^{-1}B^*Pz_{opt}(t) = -Kz_{opt}(t), \tag{2.19}$$

where P is a non-negative definite solution to the algebraic Riccati equation (ARE)

$$\langle Px, A^*y\rangle_H + \langle Ax, Py\rangle_H - \langle R^{-1}B^*Px, B^*Py\rangle_U + \langle Qx, y\rangle_H = 0, \tag{2.20}$$

for all x, y in $\mathcal{D}(A)$. Moreover, the feedback operator $K = R^{-1}B^*P$ is a bounded linear operator from H to U.

We are interested in the question of whether or not the feedback operator K can be represented by

$$[K\phi(t,.,.)](\xi) = \langle h(\xi,.,.), \phi(t,.,.)\rangle_H, \qquad \forall \phi \in H, \xi \in \Gamma \tag{2.21}$$

for some kernel $h \in L_2(\Gamma,\Omega_1) \times L_2(\Gamma,\Omega)$. If so, K is a Hilbert-Schmidt operator and the optimal control u is given by

$$\begin{aligned} u(t,\xi) &= -[Kz(t,.,.)](\xi) \\ &= -\int_{\Omega_1} h_v(\xi,r)v(t,r)rdr \\ &\quad -\int_{\Omega} h_T(\xi,r,\varphi)T(t,r,\varphi)rdrd\varphi. \end{aligned} \tag{2.22}$$

Here, $\xi \in \Gamma$, $z(t,r,\varphi) = [v(t,r) \quad T(t,r,\varphi)]^T \in H$, and
$h(\xi,r,\varphi) = [h_v(\xi,r) \quad h_T(\xi,r,\varphi)]^T \in L_2(\Gamma,\Omega_1) \times L_2(\Gamma,\Omega)$.

We use numerical methods to investigate this issue. Note that the representation (2.21)-(2.22) might be valid even if h is not in $L_2(\Gamma,\Omega_1) \times L_2(\Gamma,\Omega)$. In that case, the operator K is not Hilbert-Schmidt.

3 Numerical Results

We consider a finite element scheme to discretized the space. Here we give a brief description of the approximating problem; for more details we refer to [8].

Since the velocity depends only on the radius r and the temperature depends on the pair (r, φ), we consider one-dimensional and two-dimensional finite elements to approximate v and T, respectively. Bilinear sector elements in Ω were used for temperature interpolation and quadratic elements with uniform meshes on Ω_1 for velocity. Let $nr \geq 3$, $ns \geq 3$ denote the number of subdivisions in the radial and angular direction, respectively, and let $N = (nr, ns)$. We denote by V^N the finite dimensional approximating space, $V^N \subset V$. The approximate system in V^N is given by

$$\dot{z}^N(t) = A^N z^N(t) + f^N(z^N(t)) + B^N u^N(t), \quad t > 0, \qquad z^N(0) = z_0^N \tag{3.23}$$

where $z^N \in V^N$ and A^N, f^N and B^N are the approximate operators for A, B and f, respectively, obtained by using the interpolating functions on V^N.

The approximate LQR problem is to minimize the quadratic cost defined by

$$J^N(u^N) = \int_0^\infty \left[\int_\Omega (z^N)^T Q^N z^N r dr d\varphi + \int_\Gamma (u^N)^T R^N u^N d\varphi \right] dt \tag{3.24}$$

over all controls $u^N \in L_2([0,\infty); U^N)$, subject to the finite dimensional linear system (3.23). Let I_v^N, I_T^N, I_U^N be appropriate matrices associated with the identity operators $I_{L_2(\Omega_1)}, I_{L_2(\Omega)}$ and $I_{L_2(\Gamma)}$, respectively. The state weighting operator for the approximate LQR problem is

$$Q^N = \begin{pmatrix} Q_v^N & 0 \\ 0 & Q_T^N \end{pmatrix}, \qquad Q_v^N = q_v I_v^N, \quad Q_T^N = q_T I_T^N,$$

and the control weighting operator is $R^N = q_u I_U^N$ with q_v, q_T and q_u positive constants.

Let $P^N : V^N \to V^N$ be the approximate Riccati operator and K^N the corresponding approximate feedback operator. For each N, the approximate feedback operator is given by $K^N = \left(R^N\right)^{-1} \left(B^N\right)^* P^N : V^N \to U^N$, for the finite dimensional control problem, where U^N is the approximating finite dimensional control space, $U^N \subset U = L_2(\Gamma)$. By the Riesz Representation Theorem, we have

$$[K^N z^N(t,.,.)](\xi) = \langle h^N(\xi,.,.), z^N(t,.,.)\rangle_H \qquad \forall z^N \in V^N, \xi \in \Gamma \tag{3.25}$$

where $h^N \in L_2(\Gamma \times \Omega_1^N) \times L_2(\Gamma \times \Omega^N)$. Let

$$z^N(t,r,\varphi) = [v^N(t,r) \quad T^N(t,r,\varphi)]^T \in V^N$$

and

$$h^N(\xi,r,\varphi) = (h_v^N(\xi,r), h_T^N(\xi,r,\varphi))^T \in L_2(\Gamma\times\Omega_1^N)\times L_2(\Gamma\times\Omega^N).$$

The approximate optimal control has the form

$$\begin{aligned} u^N(t,\xi) &= -[K^N z^N(t,.,.)](\xi) \\ &= -\int_{\Omega_1^N} h_v^N(\xi,r)v^N(t,r)rdr - \int_{\Omega^N} h_T^N(\xi,r,\varphi)T^N(t,r,\varphi)rdrd\varphi. \end{aligned} \tag{3.26}$$

It can be shown [8] that for this finite element approximation scheme one has that

$$\|P^N - P\|_{\mathcal{L}(H)} \to 0 \qquad \text{as } N\to\infty$$

and

$$\|K^N - K\|_{\mathcal{L}(H)} \to 0 \qquad \text{as } N\to\infty.$$

However, it is not known if the approximate functional gains $h^N(\xi,r,\varphi)$ converge to a functional gain $h(\xi,r,\varphi)$. In particular, the existence of a function $h(\xi,r,\varphi)$ is not proven.

For simulation we consider a pipe with the same dimensions as the one used by Wang, Singer and Bau in their experiments [9]. We assume that water is flowing in a pipe having inner radius $r_1 = 1.1975in$ $(36.5cm)$ outer radius $r_2 = 1.2959in$ $(39.5cm)$.

We have solved the approximate LQR problem with different values of q_v, q_T and q_u. The results presented here correspond to $q_v = 1000, q_T = 50$ and $q_u = 10^{-4}$. We noticed that the functional gains for the approximate LQR problem have similar shapes for controls applied at the inner or outer boundary. Here, we show the results obtained by applying a control to the outer boundary $r = r_2$, only.

The kernel h_v^N depends on two variables, $\xi\in[0,2\pi)$ and $r\in(r1,r2)$. After multiplying h_v^N by $v^N(r)$ and integrating along $(r1,r2)$, we obtain the contribution of the velocity $v^N(t)$ to the control $u^N(t,\xi)$. Thus, we can think of $h_v^N(\xi,r)$ as a weight for the velocity. Analogously, $h_T^N(\xi,r,\pi)$ may be viewed as a weight for the temperature field. In this case, since h_T^N depends on three variables, we have fixed ξ to be $\xi=\pi$ for plotting purpose. However, similar plots are obtained for any $\xi\in[0,2\pi)$. The runs were based on $ns = 5nr$, with $nr = 6, 8, 10, 12$.

Figures 3.2-3.3 show the functional gains $h_v^N(\pi,r)$ $h_T^N(\pi,r,\pi)$, respectively, for $r\in[r_1,r_2]$. The plots provide numerical evidence to support the conjecture that functions $\hat{h}_v(\xi,r)$ and $\hat{h}_T(\xi,r,\varphi)$ exist and have nonzero support. Moreover, we see that $\hat{h}_v(\xi,r)$ and $\hat{h}_T(\xi,r,\varphi)$ (if they exist) are pointwise limits of $h_v^N(\xi,r)$ and $h_T^N(\xi,r,\varphi)$, respectively. It is interesting to

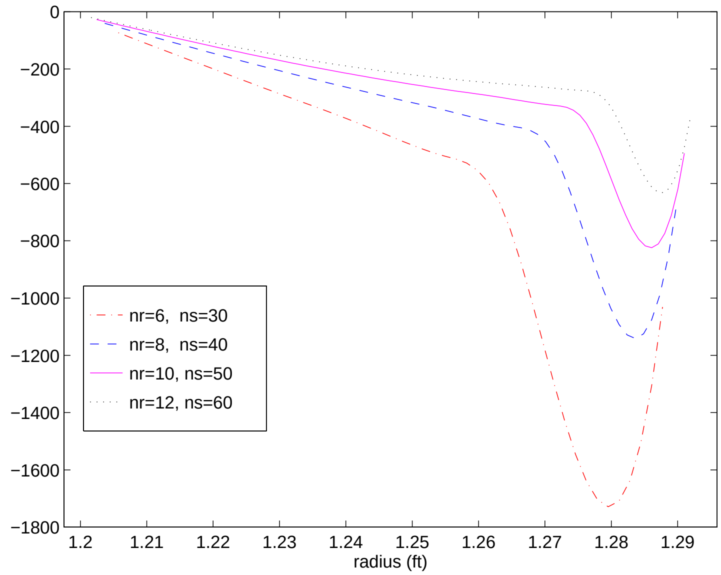

Figure 3.2: $h_v^N(\pi;r), r \in (r_1, r_2),\ N = (nr, 5nr),\ nr = 6, 8, 10, 12.$

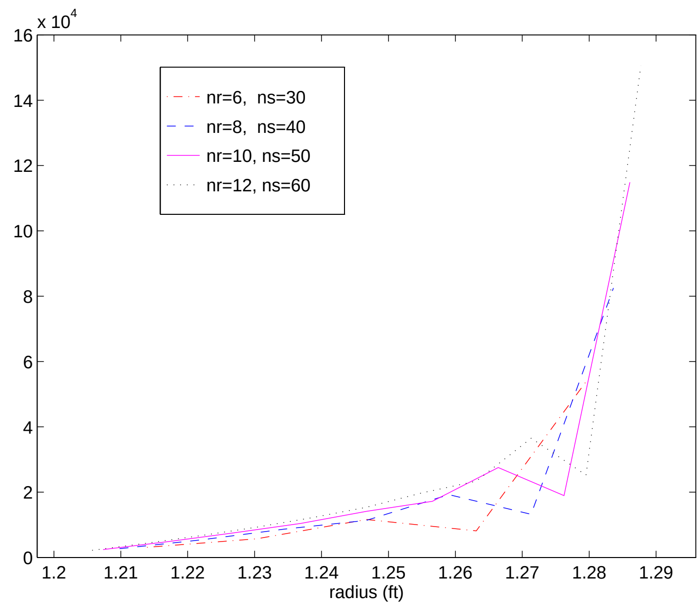

Figure 3.3: $h_T^N(\pi, r, \pi), r \in (r_1, r_2),\ N = (nr, 5nr),\ nr = 6, 8, 10, 12.$

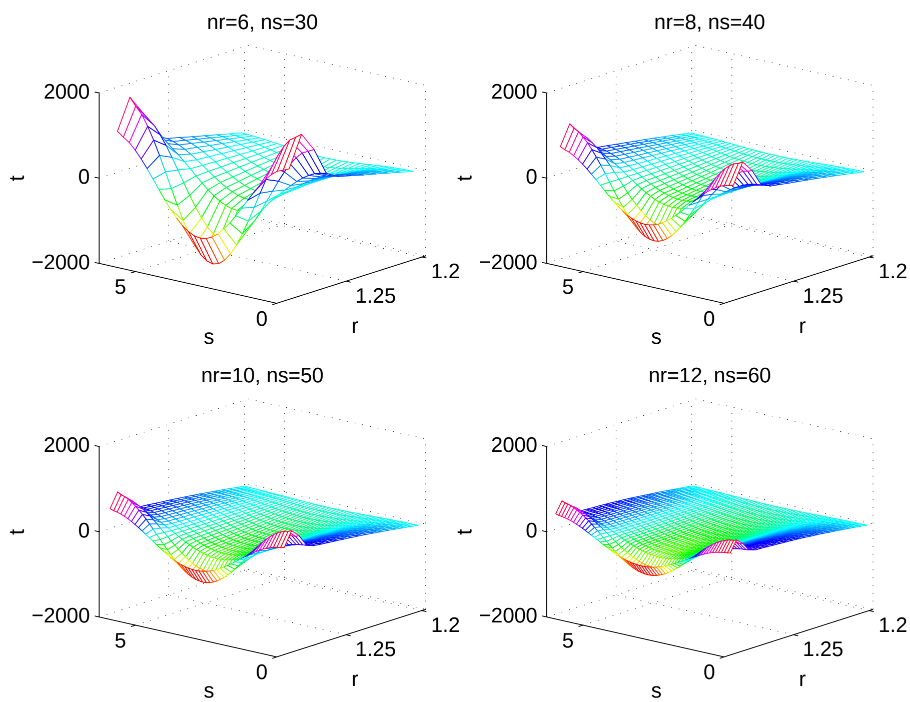

Figure 3.4: $h_v^N(\xi, r)$, $r \in (r_1, r_2)$, $N = (nr, 5nr)$, $nr = 6, 8, 10, 12$.

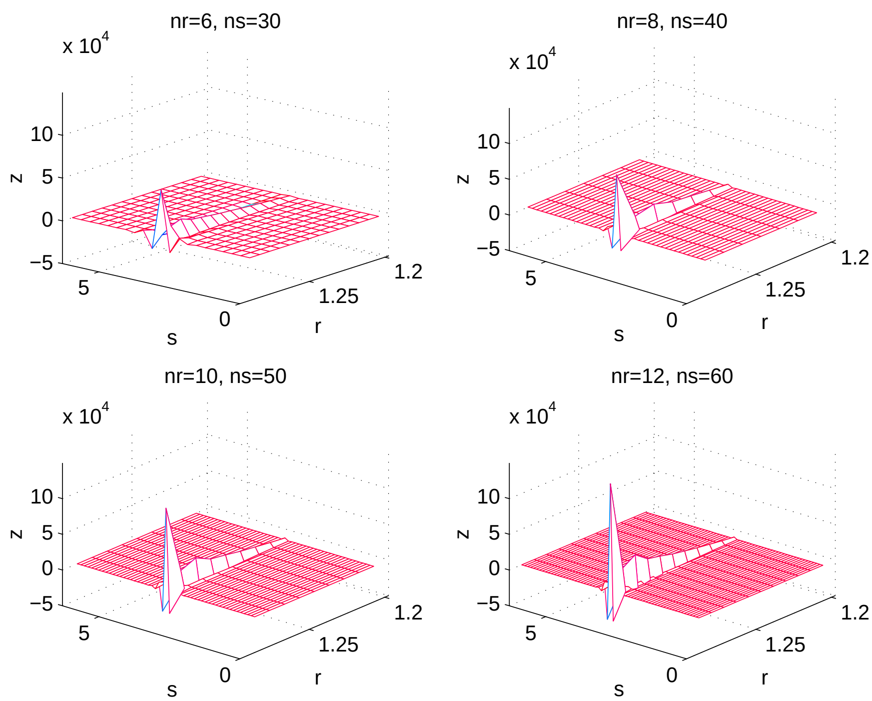

Figure 3.5: $h_T^N(\pi, r, \varphi)$, $r \in (r_1, r_2)$, $\varphi \in [0, 2\pi)$, $N = (nr, 5nr)$, $nr = 6, 8, 10, 12$.

note that the supports of the kernels h^N are concentrated near the boundary $\Gamma_2 = \{r_2\} \times [0, 2\pi)$. Hence they are almost local in space, and thus the limit, if it exists, would be almost local in space.

Figure 3.4 shows $h_v^N(\xi, r)$ with $(\xi, r) \in int(\Gamma_2 \times \Omega_1)$. The label s on one of the axis correspond to the variable ξ. Finally, in Figure 3.5 we plot the functional gains $h_T^N(\pi, r, \varphi)$, with $(r, \varphi) \in int(\Omega)$. In this graph the label s corresponds to the variable φ. Note that a singularity may to appear in the limit at $\varphi = \pi$. For different values of ξ we obtain similar plots, where the "peak" occurs at $\varphi = \xi$.

4 Conclusion

As in the case of the 2D heat equation [3], numerical results here show that $h^N(\xi, r, \varphi)$ has a nonzero compact support. Moreover, most of the support of $h^N(\xi, r, \varphi)$ is concentrated near the boundary Γ where the control is applied. Also, the kernels $h_T^N(\xi, r, \xi)$ become singular as $r \to 1.2959 = r_2$. This is true, even when different values of the parameters q_v, q_T and r are considered. However, the shape of the functional gains change slightly with these parameters.

Although the question of existence of a kernel satisfying (2.21)-(2.22) still remains open, our simulations suggest the existence of a pointwise limit $\hat{h}(\xi, r, \varphi) = (\hat{h}_v(\xi, r), \hat{h}_T(\xi, r, \varphi))^T$ of the functional gains $h^N(\xi, r, \varphi)$ on $int(\Gamma \times \Omega_1) \times int(\Gamma \times \Omega)$. In particular, we note that the numerical limit $\hat{h}(\xi, r, \varphi)$ seems to define a function with nonzero support having a singularity at (ξ, r_2, ξ).

Finally, although these numerical experiments provide considerable insight into the problem, we still need to develop a theoretical basis for answering the questions of existence and regularity.

References

[1] A. Bensoussan, G. Da Prato, M. Delfour and S. Mitter. *Representation and Control of Infinite Dimensional Systems*, **I, II**. Boston: Birkhäuser, 1992.

[2] J.A. Burns and B.B. King. A reduced basis approach to the design of low order feedback controllers for nonlinear continuous systems, *Journal of Vibration and Control*, to appear.

[3] J.A. Burns, D. Rubio and B.B. King. Regularity of feedback operators for boundary control of thermal processes, in *Proceedings of the First National Conference on Nonlinear Problems in Aviation and Aerospace*, ERAU, Daytona Beach, FL, May 1996, 67-73.

[4] H.F. Creveling, J.F. De Paz, J.Y. Baladi and R.J. Schoenhals. Stability characteristics of a single-phase free convection loop, *J. Fluid Mechanics*, **67** (1975), 65-84.

[5] A. Mertol and R. Grief. A review of natural circulation loops, *Natural Convection: Fundamentals and Applications*, (W. Aung, S. Kakac and R. Viskanta, eds.). New York: Hemisphere Press, 1985.

[6] A. Pazy. *Semigroup of Linear Operator and Applications to Partial Differential Equations.* New York: Springer-Verlag, 1983.

[7] D. Rubio. Boundary control for the chaotic flow of a thermal convection loop, *Proc. 34th IEEE Conference on Decision and Control*, New Orleans, LA, December, 1995, pp. 3734-3740.

[8] D. Rubio. *Distributed Parameter Control of Thermal Fluids*, Ph.D. Dissertation. Virginia Polytechnic Institute and State University, 1997.

[9] Y. Wang, J. Singer and H.H. Bau. Controlling Chaos in a Thermal Convection Loop, *J. Fluid Mech*, **237** (1992), 479-498.

[10] J.A. Yorke, E.D. Yorke and J. Mallet-Paret. Lorenz-like chaos in a partial differential equation for a heated fluid loop, *Physica* **24D** (1987), 279-291.

Center for Optimal Design and Control, Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0531. Currently at Center for Research in Scientific Computation, Department of Mathematics, North Carolina State University, Raleigh, NC 27695

Communicated by John Lund