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Ergodic Properties of Quotients of Horocycle Flows on the Poincaré Upper Half Plane*

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Abstract

In this paper we investigate the long term asymptotic properties of what might be called the induced horocycle flow on a compact quotient of the Poincaré upper half plane. We find that this "flow" exhibits chaotic properties in the sense that, in the long term, the area of the intersection of an open ball propagated forward by the "flow," with the original ball, tends to what would be expected if the intersections were determined in a probabilistic way.

In 1987 Doug McMahon wrote a paper on a phenomenon he named "Universal Observability," [7]. This property of flows on a manifold is defined in terms of the idea of "observability" of a dynamical system. Informally, we could say a flow is observable by a given function, F, from the manifold to the real or complex numbers, if the new function given by F composed with an orbit in the flow uniquely determines the orbit. That is, one can decide which orbit is being "observed" by F from the output of F along the trajectory of the orbit. Clearly, even with such an informal description of this property, it is easy to see, for example, that the constant functions do not observe any flow on any manifold with more than one point in it. For most flows familiar to us, it is possible to find many non-constant functions which fail to observe the flow. On the other hand, McMahon found a class of flows which were observable for *every non-constant function on the manifold*. Such flows he calls *universally observable*.

McMahon's examples draw heavily on the work of Marina Ratner who, in her papers, [8], shows that horocycle flows on certain quotients of the group $SL(2,R)$ have very dramatic ergodic properties. In particular, for

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the case where the manifold is a compact quotient of $SL(2,R)$ by a nonarithmetic subgroup, it is possible to show that the product flow is dense on the product manifold, as did del Junco and Keane, [2]. This in turn yields McMahon's result for this class of flows. We also know that universal observability implies an ergodic property somewhat weaker than that of the product flow being dense on the product manifold, [3]. Nonetheless, at this time McMahon's are the only known examples of universally observable flows.

It is worth asking why it should be so. There are two major roadblocks to extending the Ratner-McMahon example in a straightforward way. First, one might ask about cases where the quotient manifold is not compact. In this situation, there are always closed horocycles present and it is known [1] that the presence of even one closed orbit will preclude the possibility of universal observability. Closed orbits also preclude this phenomenon for geodesic flow. The second obstacle concerns the special nature of the nonarithmetic subgroups of $SL(2,R)$. The proof that "the product flow is dense on the product manifold" seems deeply intertwined with this property. This in turn precludes the construction of higher dimensional examples, because we know that for N greater than 2, nonarithmetic subgroups of $SL(N,R)$ do not exist. [6] We have shown, however, that density of the product flow is in some sense a stronger property than is necessary for universal observability, although no intermediate examples have been constructed. [3],[9]. In dimension two, all flows on all compact manifolds except the torus were shown to fail to be universally observable [1]. In this volume, however, DeStefano and Hall show that the torus can have a weak version of this property. So there are no two dimensional examples of this phenomenon.

This is perhaps an indication that McMahon's definition is too stringent to capture the properties of flows that seem very similar to the universally observable ones. In fact, the over-riding observation that has been made by many authors about the geodesic flows on these quotient manifolds is that they display very chaotic behaviour, and this is so by most definitions of chaos. The horocycle flows, on the other hand, have zero Lyapunov exponent yet exhibit "mixing," for certain cases. This paper offers a criterion based on comparing with a probabilistic event the long term effect of horocycle flows, acting on the characteristic function of a small ball. The criterion we use is a sort of local, analytic version of "mixing," so our methods have the potential to extend this type of result to a broad class of phenomena. Flows on quotients by both the arithmetic and nonarithmetic CO-compact subgroups behave similarly with respect to this criterion and it seems as though it would be possible to extend these results to non-cocompact but cofinite subgroups as well, and in fact also to other sorts of flows on these same manifolds.

Lemma 1 *Let $f(z) = \chi_A$ be the indicator function of some region A in H containing i and rotation invariant (i.e. a non-euclidean disk). Let $F(z)$ be the Poincaré series $\sum_{\gamma \in \Gamma} f(\gamma z)$ where Γ is a cocompact, discrete subgroup of $SL(2,R)$ not containing any elliptic elements. Let δ_j be the countable basis of eigenfunctions for Δ on $L^2(\Gamma \backslash H)$. Let*

$$c_j = \langle F, \delta_j \rangle$$

Then, for volume of A small enough,

$$\sum_j c_j = volume(\Gamma \backslash H)$$

Proof: This is an elementary application of the Selberg trace formula to the SO(2,**R**)-bi-invariant kernel given by F. In this context, the trace formula states:

$$\sum_j c_j = volume(\Gamma \backslash H) f(i) +$$

$$\sum_a \frac{2n(a) ln(a)}{a(a - a^{-1})} \int_{-\infty}^{\infty} f(ia^2 + x) dx.$$

For the volume of A small enough the diameter of the noneuclidean sphere doesn't reach ia^2 for the smallest value of a^2. So all of the terms in the last summation above are zero. Because $f(i)$ is 1, we obtain the result.

Lemma 2 *Let M be a fundamental region for $\Gamma \backslash H$. Let $vol(M) = v$ and, as in the lemma, let*

$$F(z) = \sum_{j=0}^{\infty} c_j \delta_j(z)$$

where $\delta_0(z) = \frac{1}{\sqrt{v}}$, which is the constant function of L^2 norm 1 on M. Then $c_0 = \frac{\sqrt{v}}{k}$ where $vol A = \frac{1}{k} vol(\Gamma \backslash H)$.

Proof:

$$c_0 = \langle F, \delta_0 \rangle$$

$$= \int_M F(z) \frac{1}{\sqrt{v}} d_\mu(z)$$

$$= \frac{1}{\sqrt{v}} \int_M F(z) d_\mu(z)$$

$$= \frac{1}{\sqrt{v}} \int_M \sum_{\gamma \in \Gamma} f(\gamma z) d_\mu(z).$$

Switching the sum and integral and doing a change of variables so that $w = \gamma z$ we get

$$= \frac{1}{\sqrt{v}} \sum_{\gamma \in \Gamma} \int_{\gamma^{-1} M} f(w) d_\mu(w)$$

$$= \frac{1}{\sqrt{v}} \int_H f(w) d_\mu(w).$$

Invoking the lemma, we get that

$$c_0 = \frac{1}{\sqrt{v}} \frac{v}{k} = \frac{\sqrt{v}}{k}.$$

Theorem *Let $f(z) =$ the characteristic function of A where, as before, A is a noneuclidean disk around i of volume $volA = \frac{1}{k} vol(\Gamma \backslash H)$. Suppose the spectrum of the Laplace-Beltrami operator on $\Gamma \backslash H$ is less than $-\frac{1}{4}$. Let Γ be a cocompact, discrete subgroup of $SL(2, R)$ not containing any elliptic elements, with some fundamental region M containing i. Let*

$$F(z) = \sum_{\gamma \in \Gamma} f(\gamma z)$$

and

$$F_{t,\theta}(z) = \sum_{\gamma \in \Gamma} f_{t,\theta}(\gamma z) = \sum_{\gamma \in \Gamma} f(\theta \gamma z + t)$$

where for ease of notation we will let θw denote the Mobius transformation which rotates w through an angle θ around i. Then, for t sufficiently large, the limit as k approaches infinity of the quantity

$$\frac{1}{\pi} \int_{\theta=0}^{\pi} \int_M F(z) F_{t,\theta}(z) d_\mu(z) = \frac{1}{k^2} vol(M)$$

That is, the average behaviour of the horocycle flow on M becomes approximately random.

Proof: Ignoring for a moment the integral in θ, we can say that

$$F(z) = \sum_{j=0}^{\infty} c_j \delta_j(z),$$

so

$$\int_M F(z) F_{t,\theta}(z) d_\mu(z)$$

$$= \int_M \sum_{j=0}^{\infty} c_j \delta_j(z) F_{t,\theta}(z) d_\mu(z).$$

By Lemma 1, the inside sum converges absolutely so this becomes

$$= \sum_{j=0}^{\infty} c_j \int_M \delta_j(z) F_{t,\theta}(z) d_\mu(z)$$

$$= c_0 \int_M \frac{1}{\sqrt{v}} F_{t,\theta}(z) d_\mu(z)$$

$$+ \sum_{j=1}^{\infty} c_j \int_M \delta_j(z) F_{t,\theta}(z) d_\mu(z).$$

Let us look at the first term of this sum. We have

$$c_0 \int_M \frac{1}{\sqrt{v}} F_{t,\theta}(z) d_\mu(z)$$

$$= \frac{\sqrt{v}}{k} \frac{1}{v} \int_M \sum_{\gamma \in \Gamma} f(\theta\gamma z + t) d_\mu(z).$$

As before, exchange the sum and integral and set $w = \gamma z$ to get

$$= \frac{1}{k} \int_H f(\theta w + t) d_\mu(w),$$

and since $u = \theta w + t$ is an isometry we have, (reinserting the integral in θ),

$$c_0 \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_M \frac{1}{\sqrt{v}} F_{t,\theta}(z) d_\mu(z)$$

$$= \frac{1}{\pi} \frac{1}{k} \int_{\theta=0}^{\pi} \int_M f(u) d_\mu(u) d\theta$$

$$= \frac{1}{k} \frac{v}{k} = \frac{v}{k^2}.$$

Now we must estimate the remaining terms. We have, for all other j a contribution of:

$$\sum_{j=1}^{\infty} c_j \frac{1}{\pi} \int_M \int_\theta \delta_j(z) F_{t,\theta}(z) d\theta d_\mu(z).$$

Using the same unwinding argument as before, this sum

$$= \sum_{j=1}^{\infty} c_j \frac{1}{\pi} \int_M \int_\theta \delta_j(z) f(\theta(z)+t) d\theta d_\mu(z).$$

Now the inside integral,

$$\int_\theta f(\theta z + t) d\theta$$

is an $SO(2)$-invariant function of z with compact support and can be approximated in L^2 by an infinitely differentiable function with the same compact support, $g(z,t)$. We can then invoke Selberg's Lemma (Hejhal, p.8) to replace the eigenfuction δ_j with the spherical eigenfuction,

$$h_{s_j}(z) = \int_\theta Im(k_\theta z)^{s_j} d\theta$$

where $s_j(s_j-1)\delta_j(z) = \Delta\delta_j(z)$.

The contribution of the remaining terms now becomes equal to

$$= \sum_{j=1}^{\infty} c_j \frac{1}{\pi} \int_M h_{s_j}(z) \int_\theta f(\theta(z)+t) d\theta d_\mu(z).$$

Letting $w = \theta z$ and integrating out θ, we have that this expression equals

$$= \sum_{j=1}^{\infty} c_j \int_M h_{s_j}(z) f(w+t) d_\mu(w).$$

Now, $s_j = \frac{1}{2} + \rho_j$ for this case (as in Hejhal,4), and $h_{s_j}(z)$ is actually a multiple of an associated Legendre function (see Terras, p. 141). This means

$$h_{s_j}(z) = P_{-s}(coshr)$$

for $z = k_u e^{-r} i$. In other words, r is the noneuclidean distance from i to z. Further,

$$P_{-s}(coshr) = \frac{1}{2\pi} \int_0^{2\pi} (z + \sqrt{z^2-1} cos(u))^{-s} du$$

and we have the asymptotic estimate (from Terras, p.144):

$$P_{-s}(coshr) \sim \frac{\Gamma(ir_j)}{\sqrt{\pi}\Gamma(\frac{1}{2}+ir_j)} (2x)^{-\frac{1}{2}+ir_j} + \frac{\Gamma(-ir_j)}{\sqrt{\pi}\Gamma(\frac{1}{2}-ir_j)} (2x)^{-\frac{1}{2}-ir_j}.$$

In addition, Lebedev (pp. 4-15) gives the estimate:

$$\Gamma(a+bi)=\sqrt{2\pi}e^{-\frac{1}{2}\pi|b|}|b|^{a-\frac{1}{2}}(1+\epsilon(a,b))$$

where, as $|b|$ approaches infinity, $\epsilon(a,b)$ goes to zero uniformly in a strip around $a=0$.

Using these facts we can bound

$$|P_{-\frac{1}{2}+i\rho_j}(x)|\leq M(2x)^{-\frac{1}{2}}$$

for large enough x. So,

$$|h_{s_j}(z)|\leq M(2cosh(r)^{-\frac{1}{2}}$$

is a uniform bound in the s_j. Using this bound we remind the reader that $f(z+t)$ has support in a disk of volume $\frac{v}{k}$ at a distance r from i. So we can choose t large enough that $|h_{s_j}(z)|\leq\epsilon$ for any ϵ. Then, our sum containing all the remaining terms is less than

$$\sum_{j=1}^{\infty}c_j\epsilon\frac{v}{k}.$$

Now, we know from Hejhal (4) that

$$\sum_{j=1}^{\infty}c_j=N$$

and is a finite sum. So, as t approaches infinity, the contribution from all of the remaining terms in the series approaches zero. **QED**

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