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A Universally Observable Flow on the Two-Dimensional Torus*

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Abstract

In this paper, we give the motivation and a brief outline of the construction of an elementary universally observable flow on the two-dimensional torus, i.e. dynamics which are observable by every continuous nonconstant real valued function on the state space. The flow in the construction is C^0 but not C^1. The ideas involved come from topological dynamics and small divisor problems.

Key words. universal observability, torus, dynamical system.

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1 Introduction

A general question in control theory is whether the solution of a dynamical system is uniquely determined by a set of measurements of the system. If this is the case, the system is said to be observable. For an overview of this problem, see [8] and [2]. To be more precise, let M be a manifold (state space),

$$\phi : M \times \mathbf{R} \to M$$

a flow on M (solution of an autonomous differential equation on M) and let $h : M \to \mathbf{R}$ be a continuous function.

Definition 1.1 *We say h* **observes** *the flow ϕ if*

$$h(\phi(x,t)) = h(\phi(y,t))$$

for all $t \geq 0$, then $x = y$.

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More concisely, h observes ϕ if the values of h along positive orbits can be used to distinguish any pair of initial conditions. In order for a flow ϕ to be observed by a particular function h, distinct points of M must have orbits which separate relative to the level sets of h. One suspects that the more orbits separate, the greater the number of functions h which observe the flow will be. The extreme case is a flow which is observable by every nonconstant continuous function.

Definition 1.2 *A flow* $\phi : M \times \mathbf{R} \to M$ *is called* **universally observable** *if it is observed by every nonconstant, continuous real valued function* $h : M \to \mathbf{R}$.

Finding universally observable flows is a nontrivial exercise. The only previously known example was found by D. McMahon [7] and is a class of three-dimensional manifolds (SL(2,$\mathbf{R}$) modulo a certain type of subgroup), with horocycle flow. This example exhibits strong ergodic and dynamical properties (see [3]).

After McMahon discovered the example of a universally observable system, several others studied the question of existence of other such flows. The results of Byrnes, Dayawansa and Martin [1], DeStefano [4] and Wallace [9] imply that the only possibility for a smooth low dimensional universally observable system is a flow on the torus with orbit structure topologically equivalent to winding lines with constant irrational slope. This work motivated the construction described below.

Recently, DeStefano and Markley [6] discovered that the notion of a universally observable flow is almost identical to the notion of prime flows in topological dynamics. In fact, if a flow is universally observable then it is prime. So the construction described below has significance in the field of topological dynamics as well.

In this note we outline the construction of a continuous, universally observable flow on the two-dimensional torus $T^2 = S^1 \times S^1$. The details of this construction can be found in [5], so here we give only the motivation and main ideas. Perhaps surprisingly, the techniques involved in the construction include number theory related to "small divisor" problems.

This construction generates a class of continuous universally observable flows which are not C^1. We end with some questions and conjectures concerning the existence of smoother universally observable flows.

2 Sufficient Conditions

The first step is to give more topological conditions on the orbits of a flow which imply that the flow is universally observable.

Definition 2.1 *A flow $\phi : M \times \mathbf{R} \to M$ is said to satisfy* **Property W** *if (i) every orbit is dense in M and (ii) for every $x, y \in M$ with y not on the orbit of x, the set $\{(\phi(x,t), \phi(y,t)) : t \geq 0\}$ is dense in $M \times M$.*

That property W implies universal observability can be informally seen as follows: Fix a nonconstant function $h : M \to \mathbf{R}$. If $x, y \in M$ are on distinct ϕ orbits then for some t, $\phi(x,t)$ is as close as we like to where h assumes its maximum while $\phi(y,t)$ is as close as we like to where h assumes its minimum value, so $h(\phi(x,t)) \neq h(\phi(y,t))$.

If x and y are on the same orbit, i.e., $y = \phi(x, t_1)$ for some t_1, the situation is more complicated. The following "folklore" theorem (whose proof was outlined for us by Nelson Markley) is precisely the necessary tool.

Lemma 2.2 *If ϕ satisfies property W then for any t_1 and any $x \in M$ the set $\{\phi(x, nt_1) : n = 0, 1, 2, ...\}$ is dense in M.*

Now, if x and y are on the same orbit, then $y = \phi(x, t_1)$ for some time t_1. So if $h(\phi(x,s)) = h(\phi(y,s))$ for all $s \geq 0$ then

$$h(\phi(x, nt_1)) = h(\phi(y, nt_1)) = h(\phi(x, (n+1)t_1))$$

for $n = 0, 1, 2, ...$ This implies that h is constant on the set $\{\phi(x, nt_1) : n = 0, 1, 2, ...\}$ which by the lemma is dense in M. This contradicts that h is nonconstant. (For the proof of the lemma see [5]).

3 Ideas of the Construction

We begin with the representation of the two-dimensional torus as the unit square with sides identified and coordinates (η, θ) (See Figure 1). Pick a rational p_1/q_1 and consider the flow given by

$$\begin{aligned} \frac{d\eta}{dt} &= \frac{p_1}{q_1} \\ \frac{d\theta}{dt} &= 1. \end{aligned}$$

This flow is not universally observable for the following two reasons:

- Orbits are periodic, not dense.
- Orbits move at a constant speed in the θ-direction, so two points starting with the same θ coordinate have the same θ-coordinate for all t.

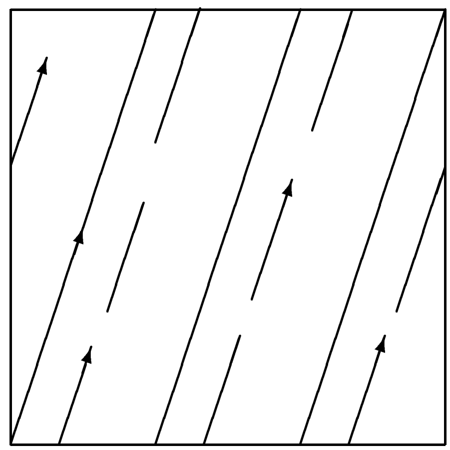

Fig. 1: Two orbits for $p_1/q_1 = 1/3$.

The first problem can be fixed by rotating the vectors in the vector field slightly so that their slope is irrational. To fix the second problem, the speeds of the orbits must be adjusted (without changing the slope) so that different orbits move with different speeds in the θ-direction.

We deal with the second problem first. We adjust the speeds of the orbits (i.e., change the length of the vectors in the vector field) so that the time required for a point $(\eta, 0)$ to cross to $\theta = 1$ is given by a piecewise linear, saw tooth function with period $1/q_1$ as in Figure 2 below.

Let $\pm s_1$ be the slope of this function, so its maximum value is $1 + s_1/(2q_1)$ and its minimum value is 1. With this adjustment to the speeds, two orbits which are a small distance ϵ apart separate in the θ-direction by $\epsilon \times s_1 \times t$ in time t. Hence, by taking t large, two points on different orbits can be made to separate in the θ-directions (i.e. lap each other on the torus) as far as we like. (Note that in the detailed construction, certain sine functions are used instead of a saw tooth function for technical reasons.)

Now we adjust the slopes of the vectors (without changing their lengths). We choose a new rational p_2/q_2 very close to p_1/q_1. We do this by considering continued fraction expansions. For example, if $p_1/q_1 = 1/a_1$ for any positive integer a_1, then let

$$\frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}}$$

where a_2 is a large positive integer. Now rotate each vector in the vector field so that it has the direction of $(p_2/q_2, 1)$.

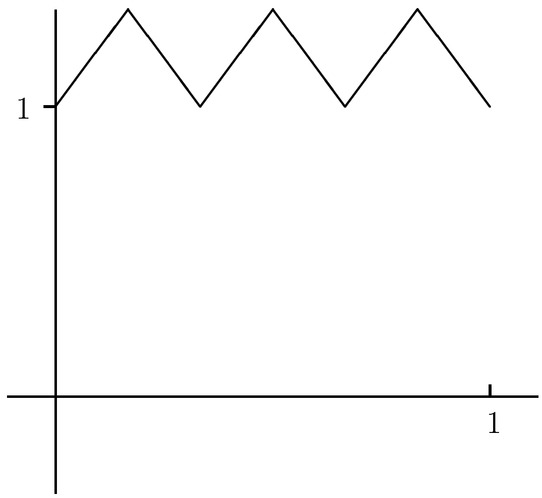

Fig. 2: Crossing times for $(\eta, 0)$.

This adjustment gives a flow for which the orbits have much longer period and are much "more dense" than the previous flow. However, it introduces some problems. As shown in Figure 3, the orbits now cover much more of the torus, so points move slowly on part of their orbits and more quickly on other parts (i.e., the "crossing time" from $\theta = 0$ to $\theta = 1$ changes along the orbit). This means that we do not have the option of waiting for arbitrarily large times t for points to separate. In fact, we can only assume that the relative speeds of two orbits is maintained for time t with $0 \leq t \leq q_2/8$. Therefore we must restrict attention to pairs of orbits at least $\epsilon = 1/q_2$ apart. For these orbits, in the allowed time we obtain a separation in the θ-direction by at least $(1/q_2) \times s_1 \times (q_2/8) = s_1/8$.

Now repeat the process. That is, first adjust the lengths of the vector fields giving a crossing time which is a piecewise linear function with slope s_2 and period $1/q_2$ (see Figure 4). Now adjust the slope to be a new rational p_3/q_3, and so forth.

At the n^{th} stage, two orbits at least $1/q_{n+1}$ apart separate in the θ-direction by at least $s_n/8$ in time $q_{n+1}/8$ where s_n is the slope of the n^{th} adjustment of the crossing times. As long as the s_n's tend to infinity as n tends to infinity, orbits in the limit flow will separate arbitrarily far in the θ-direction. The limit of the slopes of the orbits (limit of the p_n/q_n) has infinite continued fraction, hence, is irrational. So the limit flow satisfies property W.

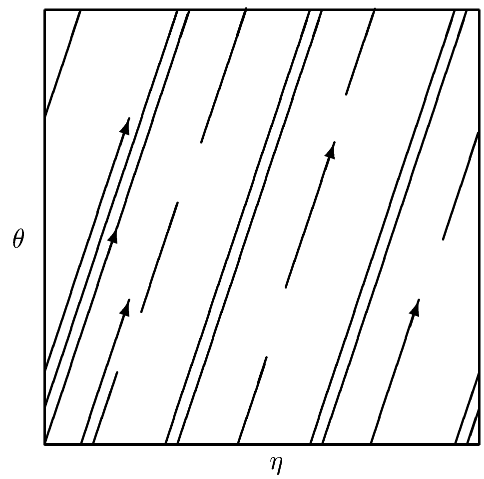

Figure 3: Two orbits for p_2/q_2.

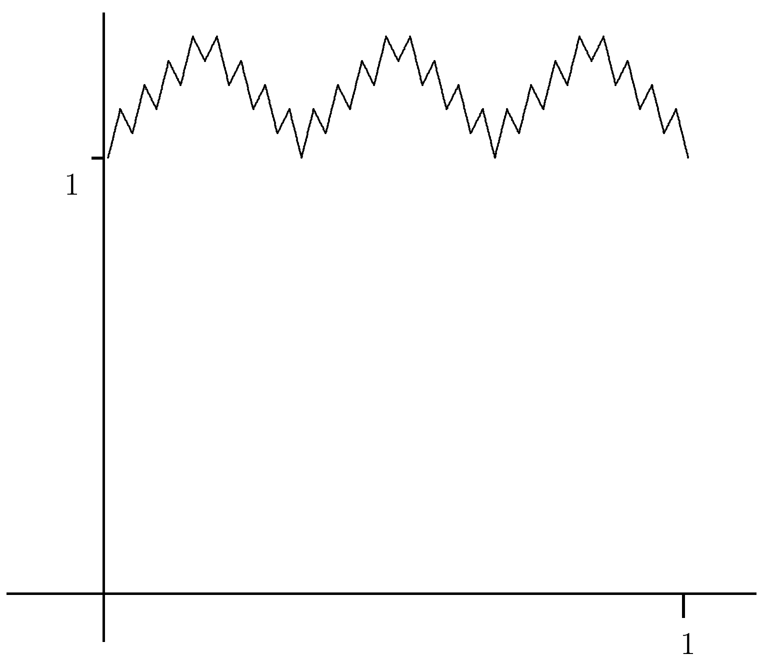

Figure 4: Crossing times for the second stage.

4 Four Details

The outline above generates many questions. In this section we deal with four major details of the construction. The fourth (convergence of the sequence of flows constructed above) is the most important and interesting.

1) Can the n^{th} step be done independently of the previous steps? At the n^{th} flow, the crossing time function is the sum of all the previous adjustments. Can the terms in the crossing time function with period $1/q_j$ for $j < n$ be ignored at the n^{th} step?

The answer is yes, provided that q_n is sufficiently large with respect to q_j for $j < n$. The period $1/q_j$ oscillations of the crossing time will average out to almost a constant over an orbit for the flow with slope p_n/q_n.

2) For the n^{th} flow, for each x there is a strip of excluded points, i.e., points y which are so close to the orbit of x that in the time available, the orbits of x and y do not separate in the θ-direction. Do these excluded strips intersect to just the orbit or x?

The answer is yes. For the n^{th} flow, the excluded strip has width in the η-direction of $1/q_{n+1}$ and length in the θ-direction of q_n. Hence, the intersection of the n^{th} and $n+1^{\text{st}}$ strips for a point x consists of a single strip of width $1/q_{n+2}$ and length q_n. Intersection with subsequent strips yields a narrower and narrower strip of bounded length around the first segment of the orbit of x, so the intersection of all the strips from some n on is contained in the orbit of x.

3) For the n^{th} flow, points on different orbits can have the same crossing time (see Figure 5). Do these orbits separate in the θ-direction?

The answer is yes. Once we have progressed to the $n+1^{\text{st}}$ flow, these two orbits have period q_{n+1}. Hence, as time t increases, the intersection points of these orbits with the line $\theta = 0$ slowly migrate along the η axis. At some time less than q_{n+1}, they are in a position where the crossing times are such that the arguments above apply and they separate in the θ-direction.

4) Does this sequence of flows converge?

The answer is yes, but just barely. For the n^{th} flow, we increase the crossing time by at most $s_n/(2{*}q_n)$. We must have the s_n tending to infinity as n increases so that orbits separate arbitrarily far in the θ-direction. We can arrange, by choosing the rationals p_n/q_n, that the q_n tend to infinity rapidly as n increases (this involves choosing the a_n's in the continued fraction expansion to be increasing rapidly). Hence, $\Sigma s_n/(2q_n)$ can be made to converge and the sequence of flows converges in the sup norm.

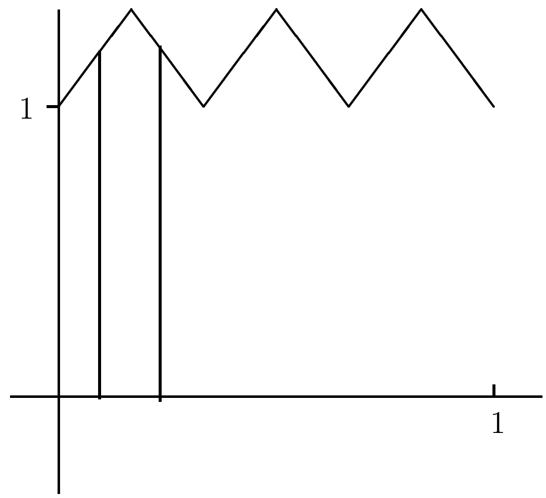

Figure 5: Two points $(\eta_1, 0)$ and $(\eta_2, 0)$ can have the same crossing time.

However, because the slopes s_n blow up, Σs_n definitely diverges. Hence the limit is not C^1.

Note that in the full construction, smooth functions are used and the resulting crossing time function converges uniformly. However, the outline above more clearly illustrates the motivation and main ideas involved.

5 Comments and Questions

An immediate questions is:

Do there exist C^k universally observable flows on the two torus for $k > 0$?

It is not uncommon in constructions of this type that the differentiability of the result can be traded against the number theory properties of the limiting slope. Preliminary calculations indicate that if the limiting slope has nice number theoretic properties (bounded entries in the continued fraction expansion) then the resulting flow could never be C^k for $k \geq 3$.

For our example, there is another obstruction to differentiability. For each flow we must choose a width for the excluded strips (orbits which are too close together to separate in the allowed time). If these strips are chosen wider, then the slope of the change in crossing time can be reduced. However, if the width of the strip is chosen too wide the intersection of infinitely many of them is more than just a single orbit.

It may be possible to constuct more differentiable flows on the torus for which almost every pair of initial conditions can be distinguished by any nonconstant continuous function. We suspect that there is an upper bound (perhaps only C^0) to truely universally observable flows on the two torus.

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