Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 3, 1998, pp. 1–25

On Observer Design for Interconnected Systems*

Gildas Besançon   Hassan Hammouri

Abstract

Following previous results on reduced observer design for nonlinear systems [1], this paper proposes Lyapunov-based conditions for observer design for interconnected systems. Examples of systems admitting observers with *partial* correction are provided, as well as systems with *full* correction. The results are illustrated an induction motor.

Key words: observers, nonlinear systems, Lyapunov function

1 Introduction

There is no systematic method to design an observer for a given nonlinear control system, but several designs are available according to the specific characteristics of the considered nonlinear system. If a system does not completely satisfy any of the known properties, it may satisfy some of them *partly*. In other words, it may be seen as an interconnection between several subsystems, where each of these subsystems satisfies some required properties for an observer to be computable. The idea is then to design an observer for the whole system, from separate designs for each subsystem, and consequently, the problem becomes analogous, to some extent, to the so-called *separation principle* for observer-based control: in that case, a controller is designed assuming that all states are available, and separately, an observer is designed assuming that the control is known. For linear systems, the achieved controller based on the estimates of the states given by the achieved observer is proved to be still stabilizing. However, this is no longer guaranteed for general nonlinear systems.

*Received January 6, 1997; received in final form August 21, 1997. Summary appeared in Volume 8, Number 3, 1998.

Our present problem is quite similar, but dealing with observer design only: for the sake of simplicity we only consider the case of two interconnected subsystems of the following general form:

$$(\Sigma)\begin{cases}\dot{x}_1 &= f_1(x_1, x_2, u)\\ \dot{x}_2 &= f_2(x_2, x_1, u)\\ y &= \begin{pmatrix} h_1(x_1)\\ h_2(x_2)\end{pmatrix} = \begin{pmatrix} y_1\\ y_2\end{pmatrix}\end{cases} \tag{1.1}$$

with $x_i \in I\!R^{n_i}, u \in I\!R^m, y_i \in I\!R^{\eta_i}, n_1 + n_2 = n, \eta_1 + \eta_2 = p$.

The idea is then to design an observer for the whole system from the "separate" synthesis of observers for each subsystem (2.2) below, assuming that for each of these separate designs, the states from the other subsystem are available.

Notice that as in the observer-based control, the observer resulting from such a design is not always stable, as illustrated by examples in the paper. Notice also that two situations may occur: either the convergence rate of the resulting observer can be *arbitrarily fast*, which happens when both of the separate synthesis allow arbitrary rate of convergence (observer with *full* correction), or the rate of convergence of the complete observer admits an upper bound, which occurs when one of the separate designs has a convergence rate imposed by the subsystem (observer with *partial* correction). In particular this second case extends previous results on reduced observer design [1].

Following these ideas, section 2 gives general Lyapunov-like *sufficient* conditions for such separate designs to be possible, highlighting the two situations which can occur, and as an illustration, section 3 proposes an observer design corresponding to the second situation for a newly considered class of systems. Section 4 then provides examples of systems for which observers can be designed with arbitrary convergence speed, and section 5 finally proposes some observer designs for an induction motor as an illustrative example of the methodology proposed in this paper. Some general conclusions are given in section 6.

2 General Problem Statement

2.1 Observer for strongly interconnected systems

By strongly interconnected systems, we mean systems of the following form:

$$(\Sigma)\begin{cases}\dot{x}_1 &= f_1(x_1, x_2, u)\\ \dot{x}_2 &= f_2(x_2, x_1, u)\\ y &= \begin{pmatrix} h_1(x_1)\\ h_2(x_2)\end{pmatrix} = \begin{pmatrix} y_1\\ y_2\end{pmatrix}\end{cases} \tag{2.1}$$

where f_1 and f_2 are $\mathcal{C}^\infty$ functions w.r.t. their arguments.

We assume that the states x_i remain in some open sets $X_i \subset \mathbb{R}^{n_i}$ and that $u \subset U \subset \mathbb{R}^m, y_i \in \mathbb{R}^{\eta_i}$, with $n_1 + n_2 = n, \eta_1 + \eta_2 = p$.

By considering models of physical processes, we can assume that inputs are bounded borelian functions and belong to some set $\mathcal{U} \subset \mathcal{L}^\infty(\mathbb{R}^+, U)$ (the space of all bounded borelian functions taking their values in U), and we denote by $\mathcal{X}_i := \mathcal{AC}(\mathbb{R}^+, \mathbb{R}^{n_i})$ the spaces of absolutely continuous functions from $\mathbb{R}^+$ into $\mathbb{R}^{n_i}$, for $i = 1, 2$.

In the sequel, when i will be an index describing $\{1, 2\}$, $\bar{\imath}$ will denote the complementary index in $\{1, 2\}$ (i.e. $\{i, \bar{\imath}\} = \{1, 2\}$). Let us then define two subsystems $(\Sigma_i), i = 1, 2$ as follows:

$$(\Sigma_i) \left\{ \begin{array}{rcl} \dot{x}_i &=& f_i(x_i, v_{\bar{\imath}}, u) \\ y_i &=& h_i(x_i) \end{array} \right. \tag{2.2}$$

where the input $(v_{\bar{\imath}}, u)$ is in $\mathcal{X}_{\bar{\imath}} \times \mathcal{U}$.

Consider also two systems $(\mathcal{O}_i)$, for $i = 1, 2$ defined by:

$$(\mathcal{O}_i) \left\{ \begin{array}{rcl} \dot{z}_i &=& f_i(z_i, v_{\bar{\imath}}, u) + k_i(g_i, z_i)(h_i(z_i) - y_i) \\ \dot{g}_i &=& G_i(z_i, v_{\bar{\imath}}, u, g_i) \end{array} \right. . \tag{2.3}$$

Their inputs are $u, v_{\bar{\imath}}$ and y_i, k_i and G_i are smooth functions w.r.t. their arguments, and the state variable (z_i, g_i) belongs to $(\mathbb{R}^{n_i} \times \mathbb{G}_i)$ where $\mathbb{G}_i$ is a subset of $\mathbb{R}^{\bar{n}_i}$ (for some integer $\bar{n}_i$), which is positively invariant by the second equation of (2.3).

In general, if for i$= 1, 2$, each system $(\mathcal{O}_i)$ is an asymptotic observer for (Σ_i), the following interconnected system

$$(\mathcal{O}) \left\{ \begin{array}{rcl} \dot{\hat{x}}_1 &=& f_1(\hat{x}_1, \hat{x}_2, u) + k_1(\hat{g}_1, \hat{x}_1)(h_1(\hat{x}_1) - y_1) \\ \dot{\hat{x}}_2 &=& f_2(\hat{x}_2, \hat{x}_1, u) + k_2(\hat{g}_2, \hat{x}_2)(h_2(\hat{x}_2) - y_2) \\ \dot{\hat{g}}_1 &=& G_1(\hat{x}_1, \hat{x}_2, u, \hat{g}_1) \\ \dot{\hat{g}}_2 &=& G_2(\hat{x}_2, \hat{x}_1, u, \hat{g}_2) \end{array} \right. \tag{2.4}$$

is not necessarily an observer for (Σ).

Counter-example 2.1 *Consider the following simple example for (Σ):*

$$\left\{ \begin{array}{rcl} \dot{x}_{11} &=& x_{12} + u x_{22} \\ \dot{x}_{12} &=& 0 \\ \dot{x}_{21} &=& x_{22} + u x_{12} \\ \dot{x}_{22} &=& 0 \\ y &=& (x_{11}, x_{21})^T \end{array} \right. \tag{2.5}$$

Clearly here, each subsystem (Σ_i) $\{$ $\dot{x}_{i1} = x_{i2} + u v_{\bar{\imath}},\ \dot{x}_{i2} = 0,\ y_i = x_{i1}$ is linear and observable, and thus admits an observer for any input.

However, one can see that for the constant input $u \equiv 1$, any initial state $x^0 = (0, x_{12}^0, 0, x_{22}^0)^T$ s.t. $x_{12}^0 = -x_{22}^0 \neq 0$ exactly gives the same output as the initial state $\underline{0} = (0,0,0,0)^T$. Consequently, since the observer only uses information from the input and the output, there is no interconnected observer of the form (2.4) which can distinguish x^0 from $\underline{0}$ for system (2.5) with input $u \equiv 1$, i.e. we cannot obtain an observer for the whole system from separate observer designs achieved for each subsystem (Σ_i).

Our purpose in this paper is thus to give sufficient conditions which ensure the convergence of the interconnected observer $(\mathcal{O})$. In other words, the problem addressed is the following one:

> *If for $i = 1,2$, $(\mathcal{O}_i)$ of equation (2.3) is an asymptotic observer for (Σ_i) of equation (2.2) (in the sense that for any initial state z_i, we have $z_i(t) - x_i(t) \to 0$ as $t \to \infty$ for any $u \in \mathcal{U}$ and $v_{\bar{\imath}} \in \mathcal{X}_{\bar{\imath}}$), then which conditions ensure that $(\mathcal{O})$ of equation (2.4) is an asymptotic observer for (Σ) of equation (2.1) for any $u \in \mathcal{U}$ and any initial state (z_1, z_2)?*

We can remark here that if the convergence of $(\mathcal{O}_i)$ is not guaranteed *for any* $v_{\bar{\imath}} \in \mathcal{AC}(\mathbb{R}^+, X_{\bar{\imath}})$, then the problem cannot be solved, since there may exist $u \in \mathcal{U}$ such that $(\mathcal{O})$ does not converge (counter-example 2.2 below).

In order to formulate a solution to the considered problem, we first set $e_i := z_i - x_i$, and for any $u \in \mathcal{U}, v_{\bar{\imath}} \in \mathcal{X}_i$ we define the following system (where $k_i^{v_{\bar{\imath}}}(t)$ denotes gain $k_i(g_i, z_i)$ defined in (2.3)):

$$(\mathcal{E}_i^{(u,v_{\bar{\imath}})}) \left\{ \begin{array}{rcl} \dot{e}_i &=& f_i(z_i, v_{\bar{\imath}}, u) - f_i(z_i - e_i, v_{\bar{\imath}}, u) \\ && + k_i^{v_{\bar{\imath}}}(t)(h_i(z_i) - h_i(z_i - e_i)) \\ \dot{z}_i &=& f_i(z_i, v_{\bar{\imath}}, u) + k_i^{v_{\bar{\imath}}}(t)(h_i(z_i) - h_i(z_i - e_i)) \\ \dot{g}_i &=& G_i(z_i, v_{\bar{\imath}}, u, g_i) \end{array} \right. . \tag{2.6}$$

We also recall that a function $v(t,e)$ is said to be a positive definite function w.r.t. e if $\forall t \geq 0, \quad v(t,e) \geq \varphi(\|e\|)$ for some strictly increasing positive function φ such that $\varphi(0) = 0$ and $\lim_{\|e\| \to \infty} \varphi(\|e\|) = +\infty$.

We now formalize our initial assumption by the following condition (which just means that $(\mathcal{O}_i)$ is an observer for (Σ_i)):

(H1) For $i = 1,2$, and for any signal $u \in \mathcal{U}, v_{\bar{\imath}} \in \mathcal{A.C}(\mathbb{R}^+, \mathbb{R}^{n_{\bar{\imath}}})$, and any initial values $(z_i^0, g_i^0) \in \mathbb{R}^{n_i} \times \mathbb{G}_i$, there exist two positive definite functions $V_i^{u,v_{\bar{\imath}},z_i^0,g_i^0}(t, e_i)$ and $W_i^{u,v_{\bar{\imath}},z_i^0,g_i^0}(e_i)$ (only denoted by $V_i(t,e_i), W_i(e_i)$ in the sequel) such that:

$$\frac{d}{dt} V_i(t, e_i(t)) \leq -W_i(e_i(t)), \tag{2.7}$$

for every trajectory $\begin{pmatrix} e_i(t) \\ z_i(t) \\ g_i(t) \end{pmatrix}$ of $(\mathcal{E}_i^{(u,v_{\bar{\imath}})})$ such that $z_i(0) = z_i^0, g_i(0) = g_i^0$.

We then introduce the required assumptions for the interconnection between $(\mathcal{O}_i)$ and $(\mathcal{O}_{\bar{\imath}})$ to be an observer:

(H2) For $i = 1, 2$, and for any $u \in \mathcal{U}, v_{\bar{\imath}} \in \mathcal{A.C}(\mathbb{R}^+, \mathbb{R}^{n_{\bar{\imath}}})$, functions $V_i(t, e_i)$, $W_i(e_i)$ associated to $u, v_{\bar{\imath}}, z_i^0, g_i^0$ satisfy:

(i) $\forall x_i \in X_i; \forall e_i \in \mathbb{R}^{n_i}; \forall t \geq 0$,

$$\frac{\partial V_i}{\partial t}(t, e_i) + \frac{\partial V_i}{\partial e_i}(t, e_i)[f_i(x_i + e_i, v_{\bar{\imath}}(t), u(t)) - f_i(x_i, v_{\bar{\imath}}(t), u(t)) + k_i^{v_{\bar{\imath}}}(t)(h_i(x_i + e_i) - h_i(x_i))] \leq -W_i(e_i)$$

(ii) $\forall x_i \in X_i; \forall x_{\bar{\imath}} \in \mathbb{R}^{n_{\bar{\imath}}}; \forall e_i \in \mathbb{R}^{n_i}; \forall e_{\bar{\imath}} \in \mathbb{R}^{n_{\bar{\imath}}}; \forall t \geq 0$,

$$\left\| \frac{\partial V_i}{\partial e_i}(t, e_i)[f_i(x_i, x_{\bar{\imath}} + e_{\bar{\imath}}, u(t)) - f_i(x_i, x_{\bar{\imath}}, u(t))] \right\| \leq \alpha_i \sqrt{W_i(e_i)} \sqrt{W_{\bar{\imath}}(e_{\bar{\imath}})},$$

for some constant $\alpha_i > 0$

(iii) $\alpha_1 + \alpha_2 < 2$.

Remark 2.1 *Condition (i) of (H2) is stronger than (H1) since inequality (2.7) of (H1) coincides with the one in condition (H2)-(i), only along trajectories. For this reason, in the sequel we only need assumption (H2).*

The above assumptions provide the following result:

Proposition 2.1 *If assumption (H2) is satisfied, then system $(\mathcal{O})$ described by (2.4) is an asymptotic observer for system (Σ) described by (2.1).*

Proof: Set $e_i := \hat{x}_i - x_i$, for $i = 1$ to 2, and define the following systems:

$$(\mathcal{E}_i^{(u,\hat{x}_{\bar{\imath}})}) \quad \begin{array}{rcll} \dot{e}_i &=& f_i(\hat{x}_i, \hat{x}_{\bar{\imath}}, u) - f_i(\hat{x}_i - e_i, \hat{x}_{\bar{\imath}} - e_{\bar{\imath}}, u) & (2.8) \\ && + k_i^{\hat{x}_{\bar{\imath}}}(t)(h_i(\hat{x}_i) - h_i(\hat{x}_i - e_i)). & \\ \dot{\hat{x}}_i &=& f_i(\hat{x}_i, \hat{x}_{\bar{\imath}}, u) + k_i^{\hat{x}_{\bar{\imath}}}(t)(h_i(\hat{x}_i) - h_i(\hat{x}_i - e_i)) & (2.9) \\ \dot{\hat{g}}_i &=& G_i(\hat{x}_i, \hat{x}_{\bar{\imath}}, u, \hat{g}_i). & (2.10) \end{array}$$

Given $\hat{x}_1(0), \hat{x}_2(0), \hat{g}_1(0), \hat{g}_2(0)$, equation (2.8) becomes a time-varying nonlinear equation. We then define $\hat{V}_i$ by function V_i obtained with $v_{\bar{\imath}}(.) =$

$\hat{x}_{\bar{\imath}}(.)$, and we show that $\hat{V} := \hat{V}_1 + \hat{V}_2$ is a Lyapunov function for equations (2.8) for $i = 1, 2$. To that end, note that:

$$\dot{e}_i = f_i(\hat{x}_i, \hat{x}_{\bar{\imath}}, u) \quad - f_i(x_i, \hat{x}_{\bar{\imath}}, u) + k_i^{\hat{x}_{\bar{\imath}}}(t)(h_i(\hat{x}_i) - h_i(x_i)) \\ + f_i(x_i, \hat{x}_{\bar{\imath}}, u) - f_i(x_i, x_{\bar{\imath}}, u).$$

Hence

$$\begin{aligned} \dot{\hat{V}}_i(t, e_i) &= \frac{\partial \hat{V}_i}{\partial t}(t, e_i) + \frac{\partial \hat{V}_i}{\partial e_i}(t, e_i)[f_i(x_i + e_i, \hat{x}_{\bar{\imath}}, u) - f_i(x_i, \hat{x}_{\bar{\imath}}, u) \\ &\qquad + k_i^{\hat{x}_{\bar{\imath}}}(t)(h_i(\hat{x}_i) - h_i(x_i))] \\ &\quad + \frac{\partial \hat{V}_i}{\partial e_i}(t, e_i)[f_i(x_i, x_{\bar{\imath}} + e_{\bar{\imath}}, u) - f_i(x_i, x_{\bar{\imath}}, u)] \\ &\leq -\hat{W}_i(e_i) + \alpha_i \sqrt{\hat{W}_i(e_i)} \sqrt{\hat{W}_{\bar{\imath}}(e_{\bar{\imath}})} \end{aligned}$$

where $\hat{W}_i$ is defined by W_i for $v_{\bar{\imath}}(.) = \hat{x}_i(.)$. It follows that:

$$\begin{aligned} \dot{\hat{V}} &\leq \textstyle\sum_{i=1}^{2} (-\hat{W}_i(e_i) + \alpha_i \sqrt{\hat{W}_i(e_i)} \sqrt{\hat{W}_{\bar{\imath}}(e_{\bar{\imath}})}) \\ &\leq -\tfrac{1}{2}(1 - \tfrac{\alpha_1 + \alpha_2}{2})(\sqrt{\hat{W}_1(e_1)} + \sqrt{\hat{W}_2(e_2)})^2, \end{aligned} \tag{2.11}$$

which ends the proof. □

Now, coming back to the assumption of convergence of $(\mathcal{O}_i)$ *for any* $v_{\bar{\imath}} \in \mathcal{AC}(I\!R^+, X_{\bar{\imath}})$, we can show its importance on the following example:

Counter-example 2.2 *Consider the system defined by:*

$$\left\{ \begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= \begin{pmatrix} 0 & x_1 - \frac{1}{2} \\ 0 & 0 \end{pmatrix} x_2 = A(x_1) x_2 \\ y_1 &= h_1(x) = 0 \\ y_2 &= h_2(x) = C x_2 = (1 \quad 0) x_2 \end{aligned} \right. \tag{2.12}$$

and the associate subsystems:

$$(\Sigma_1) \quad \dot{x}_1 = -x_1 + u, \ y_1 = 0; \quad (\Sigma_2) \quad \dot{x}_2 = A(v_1) x_2, \ y_2 = C x_2. \tag{2.13}$$

We then design the following observers:

$$(\mathcal{O}_1) \left\{ \ \dot{\hat{x}}_1 = -\hat{x}_1 + u \right. \tag{2.14}$$

$$(\mathcal{O}_2) \left\{ \begin{aligned} \dot{\hat{x}}_2 &= \begin{pmatrix} 0 & v_1 - \frac{1}{2} \\ 0 & 0 \end{pmatrix} \hat{x}_2 - S^{-1} C^T (C \hat{x}_2 - y_2) \\ \dot{S} &= -\theta S - A^T(v_1) S - S A(v_1) + C^T C, \end{aligned} \right. , \tag{2.15}$$

where $(\mathcal{O}_1)$ *is an exponential observer for* (Σ_1) *which works for every* u*, and* $(\mathcal{O}_2)$ *is an exponential observer for* (Σ_2) *which works for every regularly persistent input* v_1 *[2].*

Clearly, functions $V_1(e_1) := e_1^2$ *and* $V_2(t, e_2) := e_2^T S_2 e_2$ *are appropriate positive definite functions for (H1) and (H2), except that* $(\mathcal{O}_2)$ *does not converge for any* v_1*. In particular,* $(\mathcal{O}_2)$ *does not converge for any input* v_1 *such that* $v_1(t)$ *exponentially tends to* $\frac{1}{2}$ *as* t *tends to infinity.*

Hence

$$\left\{\begin{array}{lcl} \dot{\hat{x}}_1 & = & -\hat{x}_1 + u \\ \dot{\hat{x}}_2 & = & \begin{pmatrix} 0 & \hat{x}_1 - \frac{1}{2} \\ 0 & 0 \end{pmatrix} \hat{x}_2 - S^{-1} C^T (C\hat{x}_2 - y_2) \\ \dot{S} & = & -\theta S - A^T(\hat{x}_1) S - S A(\hat{x}_1) + C^T C \end{array}\right. \tag{2.16}$$

does not converge for $u = \frac{1}{2}$*.*

Remark 2.2 *If functions* V_i *and* W_i *are quadratically bounded* $(\beta_{i1}\|e_i\|^2 \leq V_i(t, e_i), W_i(e_i) \leq \beta_{i2}\|e_i\|^2)$*, then the convergence is exponential.*

In view of the proof, the method can be easily extended to more than two subsystems, provided that the Lyapunov functions associated to each observer satisfy interconnected conditions like in (H2), allowing to choose their sum as a Lyapunov function for the whole system, and to bound its time derivative by some positive definite expression.

Finally, notice that if the subdivisions of the considered system (Σ) do not appear in first approach, coordinates transformations or/and output transformations can be tried to make appropriate subsystems to appear. This is what was inspected for instance in the case of cascade interconnections [3, 4, 5] for arbitrarily fast observation, or some cases of reduced order observation [1].

2.2 Observer for weakly interconnected systems

These systems take the following form:

$$(\Sigma') \left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(x_1, u) \\ \dot{x}_2 & = & f_2(x_2, x_1, u) \\ y & = & \begin{pmatrix} h_1(x_1) \\ h_2(x_2) \end{pmatrix} \end{array}\right. \tag{2.17}$$

Denoting by (Σ_i') the resulting subsystems as defined by (2.2), assumptions for the existence of an observer for (Σ') based on observers for (Σ_i') as in (2.4) can be weakened in the following way:

(H1') We assume that (Σ'_1) admits an observer of the form:

$$(\mathcal{O}'_1)\left\{\begin{array}{lcl}\dot{z}_1 & = & f_1(z_1,u)+k_1(g_1,z_1)(h_1(z_1)-y_1)\\ \dot{g}_1 & = & G_1(z_1,u,g_1)\end{array}\right. \tag{2.18}$$

and that for every $u \in \mathcal{U}$ and every admissible trajectory $x_1(t)$ of (Σ'_1) associated to u,we have $\lim_{t\to\infty} e_1(t)=0$ (with $e_1 := z_1 - x_1$) and

$$\int_0^{+\infty} \|e_1(t)\|dt < +\infty. \tag{2.19}$$

Note that this condition (2.19) is indispensable (see counter-example 2.3 below), and is satisfied as soon as $(\mathcal{O}'_1)$ is an exponential observer.

(H2') $\exists c>0;\ \forall u\in U;\ \forall x_2\in X_2, \|f_2(x_2,x_1,u)-f_2(x_2,x'_1,u)\|\leq c\|x_1-x'_1\|.$

Remark 2.3 *Since f_2 is of class $\mathcal{C}^\infty$, (H2') is always satisfied if X_1, X_2, U are bounded sets.*

(H3') (Σ'_2) admits an observer:

$$(\mathcal{O}'_2)\left\{\begin{array}{lcl}\dot{z}_2 & = & f_2(z_2,v_1,u)+k_2(g_2,z_2)(h_2(z_2)-y_2)\\ \dot{g}_2 & = & G_2(z_2,v_1,u,g_2)\end{array}\right. \tag{2.20}$$

in the sense that, by considering the associate system $(\mathcal{E}_2^{(u,v_1)})$ defined by (2.6), for any signal $u\in\mathcal{U}, v_1\in\mathcal{A.C}(I\!R^+, I\!R^{n_1})$, any initial value z_2^0, g_2^0, there exist positive definite functions $v(t,e_2)$ and $w(e_2)$ such that for every trajectory of $(\mathcal{E}_2^{(u,v_1)})$ such that $z_2(0)=z_2^0, g_2(0)=g_2^0$:

$$\frac{d}{dt}v(t,e_2(t))\leq -w(e_2(t)).$$

(H4') For any $u\in\mathcal{U}, v_1\in\mathcal{A.C}(I\!R^+, I\!R^{n_1})$, there exists a positive definite function $\bar{w}(e)$, satisfying, together with $v(t,e), w(e)$, the following:

(i) $\forall x_2\in X_2, e_2\in I\!R^{n_2}, t\geq 0,$

$$\frac{\partial v}{\partial t}(t,e_2)+\frac{\partial v}{\partial e_2}(t,e_2)[f_2(x_2+e_2,v_1(t),u(t))-f_2(x_2,v_1(t),u(t)) \\ +k_2^{v_1}(t)(h_2(x_2+e_2)-h_2(x_2))]\leq -w(e_2)$$

(ii) $\forall e_2\in I\!R^{n_2}, t\geq 0;\ v(t,e_2)\geq \bar{w}(e_2)$

(iii) $\forall e_2\in I\!R^{n_2}\backslash\mathcal{B}(0,r), t\geq 0;\ \left\|\frac{\partial v}{\partial e_2}(t,e_2(t))\right\|\leq\lambda(1+v(t,e_2(t)))$ for some constants $\lambda, r>0$ and $\mathcal{B}(0,r):=\{e_2:\|e_2\|\leq r\}$.

Note that condition (iii) is satisfied e.g. by quadratic functions $x^T R(t)x$ for some symmetric positive definite $R(t)$.

Remark 2.4 *As in the previous subsection, we can remark that condition (H4')-(i) contains in particular condition (H3'), i.e., the fact that $(\mathcal{O}_2')$ is an asymptotic observer for (Σ_2').*

Proposition 2.2 *Under assumptions (H1'), (H2') and (H4'), system (2.21) below is an asymptotic observer for (Σ') described by (2.17):*

$$\begin{array}{rcl}\dot{\hat{x}}_1 & = & f_1(\hat{x}_1,u)+k_1(\hat{g}_1,\hat{x}_1)(h_1(\hat{x}_1)-h_1(x_1))\\ \dot{\hat{x}}_2 & = & f_2(\hat{x}_1,\hat{x}_2,u)+k_2(g_2,\hat{x}_2)(h_2(\hat{x}_2)-h_2(x_2))\\ \dot{\hat{g}}_1 & = & G_1(\hat{x}_1,u,\hat{g}_1)\\ \dot{\hat{g}}_2 & = & G_2(\hat{x}_2,\hat{x}_1,u,\hat{g}_1).\end{array} \tag{2.21}$$

Remark 2.5 *By remark 2.3, assumption (H2') can be dropped if the input $u(.)$ and the unknown trajectory are bounded.*

Proof: We must prove convergence of $(\mathcal{O}_2')$. with notation $k_i^{v_i}$ of (2.6):

$$\begin{array}{rcl}\dot{e}_2 & = & f_2(x_2+e_2,x_1+e_1,u)-f_2(x_2,x_1,u)+k_2^{\hat{x}_1}(t)(h_2(\hat{x}_2)-h_2(x_2))\\ & = & f_2(x_2+e_2,\hat{x}_1,u)-f_2(x_2,\hat{x}_1,u)+k_2^{\hat{x}_1}(t)(x_2,e_2,t)\\ & & +f_2(x_2,x_1+e_1,u)-f_2(x_2,x_1,u)\end{array} \tag{2.22}$$

and we consider functions $\hat{v}(e_2,t),\hat{w}(e_2),\hat{\bar{w}}(e_2)$ defined by $v(e_2,t),w(e_2)$, $\bar{w}(e_2)$ with $v_1(.)=\hat{x}_1(.)$. Then, by condition (H4')-(i),

$$\begin{array}{rcl}\dot{\hat{v}}(t,e_2) & = & \dfrac{\partial\hat{v}}{\partial t}+\dfrac{\partial\hat{v}}{\partial e_2}\dot{e}_2\\ & \le & -\hat{w}(e_2)+\dfrac{\partial\hat{v}}{\partial e_2}(f_2(x_2,x_1+e_1,u)-f_2(x_2,x_1,u)).\end{array} \tag{2.23}$$

By (H2'), it results

$$\dot{\hat{v}}(t,e_2)\le-\hat{w}(e_2)+c\frac{\partial\hat{v}}{\partial e_2}\|e_1\| \tag{2.24}$$

and by (H4')-(iii), for $\|e_2\|>r$,

$$\dot{\hat{v}}(t,e_2)\le-\hat{w}(e_2)+c\lambda(1+\hat{v}(t,e_2))\|e_1\|\le c\lambda(1+\hat{v}(t,e_2))\|e_1\|.$$

By integrating this inequality, we get

$$Log(1+\hat{v}(t,e_2))\le c\lambda\int_0^\infty\|e_1(\tau)\|d\tau$$

and using condition (2.19) of (H1') we conclude that e_2 is bounded.

Going back to equation (2.24), it follows

$$\dot{\hat{v}}(t,e_2) \leq -\hat{w}(e_2) + \gamma\|e_1\| \quad \text{for some } \gamma > 0,$$

and we can then prove convergence of e_2 to zero: let $(\rho_i)_i$ be a sequence of real numbers such that $\rho_i > \rho_{i+1} > \ldots > 0$ and $lim_{i\to\infty}\rho_i = 0$, and set $\mathcal{B}(0,\rho_i) := \{e_2 : \hat{\bar{w}}(e_2) \leq \rho_i\}$.

Choose ρ_0 such that $\mathcal{B}(0,\rho_0)$ contains the bounded domain of e_2, and for $i \geq 0$, set $\eta_i := inf\{\hat{w}(e_2); e_2 \in \mathcal{B}(0,\rho_i)\backslash\mathcal{B}(0,\rho_{i+1})\}$. Since $\|e_1(t)\|$ tends to zero, there exists $\bar{t}_0 \geq 0$ such that $\gamma\|e_1(t)\| \leq \frac{\eta_0}{2}$ for $t \geq \bar{t}_0$. Hence, as long as $e_2 \in \mathcal{B}(0,\rho_0)\backslash\mathcal{B}(0,\rho_1)$, and $t \geq \bar{t}_0$, we have $\dot{\hat{v}}(t,e_2) \leq -\frac{\eta_0}{2} < 0$. As a result $\hat{v}(t,e_2)$ decreases, until e_2 lies into $\mathcal{B}(0,\rho_1)$ at a time t_0, and by construction, cannot go out $\mathcal{B}(0,\rho_1)$. By induction, for $i \geq 1$, if $e_2(t) \in \mathcal{B}(0,\rho_i)\backslash\mathcal{B}(0,\rho_{i+1})$, then $\exists\bar{t}_i \geq t_{i-1}$ such that for $t \geq \bar{t}_i$, $\gamma\|e_1(t)\| \leq \frac{\eta_i}{2}$. In a similar way as in the first step, $\dot{\hat{v}}(t,e_2) \leq -\frac{\eta_i}{2} < 0$ as long as $e_2 \in \mathcal{B}(0,\rho_i)\backslash\mathcal{B}(0,\rho_{i+1})$. Thus $\hat{v}(t,e_2)$ decreases until e_2 lies into $\mathcal{B}(0,\rho_{i+1})\backslash\mathcal{B}(0,\rho_{i+2})$ at time t_i. As above, we have $\forall t \geq t_i$, $e_2(t) \in \mathcal{B}(0,\rho_{i+1})$. Since $t_i \to \infty$ and $\mathcal{B}(0,\rho_i)\backslash\mathcal{B}(0,\rho_{i+1}) \to_{i\to\infty} \{0\}$, we get $e_2 \to_{t\to\infty} 0$. □

The importance of condition (2.19) can be illustrated by the following:

Counter-example 2.3 *Consider the system:*

$$(\Sigma')\left\{\begin{array}{lcl} \dot{x}_1 & = & -\frac{1}{2(1+t)}x_1 \\ \dot{x}_2 & = & -\frac{1}{4(1+t)}x_2 + x_1 \\ y & = & (0,\,0)^T \end{array}\right. \tag{2.25}$$

(Σ'_1) and (Σ'_2) clearly admit the following respective observers

$$(\mathcal{O}'_1)\ \dot{z}_1 = -\frac{1}{2(1+t)}z_1 \quad \text{and} \quad (\mathcal{O}'_2)\ \dot{z}_2 = -\frac{1}{4(1+t)}z_2 + v_1$$

for any v_1.

Moreover (H2') is clearly satisfied, and dynamics of $e_2 := z_2 - x_2$ admit $v(t,e_2) := (1+t)e_2^8$ as a Lyapunov function such that (H4') is satisfied. However $e_1(t) = \frac{e_{10}}{\sqrt{1+t}}$ and thus condition (2.19) is not verified.

It can then be easily checked that the interconnected system

$$\left\{\begin{array}{lcl} \dot{\hat{x}}_1 & = & -\frac{1}{2(1+t)}\hat{x}_1 \\ \dot{\hat{x}}_2 & = & -\frac{1}{4(1+t)}\hat{x}_2 + \hat{x}_1 \end{array}\right. \tag{2.26}$$

is not an observer for (Σ'), since $\|\hat{x}_2(t) - x_2(t)\| \to_{t\to\infty} \infty$.

Notice again that with stronger conditions, one can ensure arbitrary exponential convergence (as in [5] for block-state affine systems for instance):

Proposition 2.3 *If ($\mathcal{O}_1'$) is an exponential observer for (Σ_1'), (H2') is satisfied, and (H4') is replaced by the assumption that for any $u \in \mathcal{U}, v_1 \in \mathcal{A.C}(I\!R^+, I\!R^{n_1})$, there exist real numbers $\gamma_1, \gamma_2, \alpha, \beta_0 > 0$ such that for any $\beta > \beta_0$, $v(t, e_2)$ satisfies:*

(i) Condition (H4')-(i) where $w(e_2)$ is replaced by $\beta v(t, e_2)$.

(ii) $\gamma_1 \|e_2\|^2 \leq v(t, e_2) \leq \gamma_2 \|e_2\|^2$.

(iii) $\forall e_2 \in I\!R^{n_2}, t \geq 0;\ \left\| \frac{\partial v}{\partial e_2}(t, e_2) \right\| \leq \alpha \|e_2\|$.

Then system (2.21) is an exponential observer for (Σ_c).

For the proof, just remark that equation (2.24) in the proof of proposition 2.2 still holds with $\hat{w}(e_2)$ replaced by $\beta\hat{v}(t, e_2)$, and that new conditions (ii) and (iii) leads to:

$$\dot{\hat{v}}(t, e_2) \leq -\beta \hat{v}(t, e_2) + \delta \sqrt{\hat{v}(t, e_2)} \|e_1\| \quad \text{for some } \delta > 0.$$

Dividing this equation by $\sqrt{\hat{v}(t, e_2)}$, and using the assumption of exponential convergence of ($\mathcal{O}_1'$) (i.e., $\|e_1(t)\| \leq \rho_1 e^{-\zeta t}$, for some $\rho_1, \zeta > 0$), it follows

$$\frac{d}{dt}\sqrt{\hat{v}(t, e_2)} \leq -\beta \sqrt{\hat{v}(t, e_2)} + \bar{\delta} e^{-\zeta t}.$$

By integrating this equation, it finally results in

$$\sqrt{\hat{v}(t, e_2)} \leq \rho_2 e^{-\frac{\beta}{2} t} + \frac{\bar{\delta}}{\beta - 2\zeta} e^{-\zeta t}, \quad \text{for some } \rho_2, \bar{\delta} > 0$$

which allows us to conclude.

In particular, the convergence of the resulting full observer can be given an arbitrary exponential rate as soon as this is the case for ($\mathcal{O}_1'$).

3 Observer with Partial Correction for a Class of Nonlinear Systems

On the basis of propositions of section 2, this section presents an observer for a class of systems extending previous results of [1]. The basic idea in [1] can be summarized as follows:

Given a nonlinear system:

$$(\Sigma) \left\{ \begin{array}{rcl} \dot{x} & = & f(x, u) \\ y & = & h(x) \end{array} \right. \tag{3.1}$$

with $x \in \mathbb{R}^n$, $f \in \mathcal{C}^\infty$ w.r.t. (x,u), $y \in \mathbb{R}^p$, let us consider a change of coordinates $z = T(x) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with $z_1 = h(x)$.

System (3.1) then becomes:

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & \bar{f}_1(z_1, z_2, u) \\ \dot{z}_2 & = & \bar{f}_2(z_2, z_1, u) \\ y & = & z_1. \end{array} \right. \tag{3.2}$$

Assume that system:

$$(\Sigma_2) \quad \dot{z}_2 = \bar{f}_2(z_2, v_1, u),\ u \in \mathcal{U} \subset \mathcal{L}^\infty(\mathbb{R}^+, U), v_1 \in \mathcal{AC}(\mathbb{R}^+, \mathbb{R}^p) \tag{3.3}$$

satisfies the following condition:

$$\forall (u, v_1) \in \mathcal{U} \times \mathcal{AC}(\mathbb{R}^+, \mathbb{R}^p);\ \forall z_2^0 \neq \bar{z}_2^0,\ \|z_2^0(t) - \bar{z}_2^0(t)\| \to_{t\to\infty} 0, \tag{3.4}$$

where $z_2^0(t)$ and $\bar{z}_2^0(t)$ are resp. trajectories of (Σ_2) associated to (u, v_1) with $z_2(0) = z_2^0, \bar{z}_2(0) = \bar{z}_2^0$. Then the system:

$$\left\{ \begin{array}{lcl} \dot{\hat{z}}_2 & = & \bar{f}_2(\hat{z}_2, y, u) \\ \hat{x} & = & T^{-1} \begin{pmatrix} y \\ \hat{z}_2 \end{pmatrix} \end{array} \right. \tag{3.5}$$

is an asymptotic observer for (Σ). In [1], sufficient conditions are given in order to ensure condition (3.4) in the case when $\bar{f}_2(z_2, z_1, u) = A_2 z_2 + \varphi_2(u, z)$.

In this section, using proposition 2.1, we extend this kind of result to a class of systems of the form (3.2), where z_1 is not fully measured, but can be estimated by an observer, while z_2 still satisfies a condition like (3.4). Provided indeed, that the rate of convergence of the estimation of z_1 can be chosen high enough, then an observer can be obtained for the whole system as in previous section. As long as z_2 does not contain any measure (the output function h_2 of previous section vanishes), this situation corresponds to a whole observer for which the convergence speed cannot be arbitrarily chosen.

Here we discuss such properties for a system of the following form:

$$(\Sigma_\xi) \left\{ \begin{array}{lcl} \dot{\xi}_1 & = & A_1\xi_1 + \varphi_1(\xi_1, \xi_2, u),\ \xi_1 \in \mathbb{R}^{n_1}, u \in \mathbb{R}^m \\ \dot{\xi}_2 & = & A_2\xi_2 + \varphi_2(\xi_2, \xi_1, u),\ \xi_2 \in \mathbb{R}^{n_2} \\ y & = & C\xi_1,\ y \in \mathbb{R}, \end{array} \right. \tag{3.6}$$

with

$$\xi_1 = \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1n_1} \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

$$C = (1\ 0\ \dots\ 0),$$
$$\varphi_1(\xi_1, \xi_2, u) = \begin{pmatrix} \varphi_{11}(\xi_{11}, u) \\ \vdots \\ \varphi_{1n_1-1}(\xi_{11}, \dots \xi_{1n_1-1}, u) \\ \varphi_{1n_1}(\xi_1, \xi_2, u) \end{pmatrix}.$$

Moreover, we will need the following assumptions:

Assumption 1 *φ_1 and φ_2 are global Lipschitz functions w.r.t. ξ_1 (resp. w.r.t. ξ_2) locally uniformly w.r.t. u and globally uniformly w.r.t. ξ_2 (resp. locally uniformly w.r.t. u and globally uniformly w.r.t. ξ_1), and denote by κ_1, κ_2 (resp. $\bar{\kappa}_1, \bar{\kappa}_2$) their corresponding constants.*

This assumption can be omitted as soon as the state $(\xi_1(t), \xi_2(t))$ lies in a compact set (one can indeed, as in [6], prolongate nonlinearities φ_1, φ_2 outside a compact set to make them global Lipschitz). Moreover, considering the following subsystem:

$$(\Sigma_{\xi_1}) \left\{ \begin{array}{rcl} \dot{\xi}_1 & = & A_1\xi_1 + \varphi_1(\xi_1, v_2, u), \\ y & = & C\xi_1, \qquad (v_2, u) \in \mathcal{AC}(I\!R^+, I\!R^{n_2}) \times \mathcal{U}, \end{array} \right. \tag{3.7}$$

assumption 1—together with the structure of A_1, C, φ_1—ensures the existence of a *high gain observer* for (3.7) of the following form [6]:

$$(\mathcal{O}_{\xi_1}) \left\{ \begin{array}{rcl} \dot{\hat{\xi}}_1 & = & A_1\hat{\xi}_1 + \varphi_1(\hat{\xi}_1, v_2, u) - S_\theta^{-1}C^T(C\hat{\xi}_1 - y) \\ 0 & = & \theta S_\theta + A_1^T S_\theta + S_\theta A_1 - C^T C. \end{array} \right. \tag{3.8}$$

Assumption 2 *There exist symmetric positive definite matrices P, Q s.t.*

$$PA_2 + A_2^T P = -Q \quad \text{and} \quad \frac{\lambda_{min} Q}{2\lambda_{max} P} > \bar{\kappa}_2, \tag{3.9}$$

where λ_{min} (resp. λ_{max}) denotes the smallest (resp. largest) eigenvalue, and $\bar{\kappa}_2$ is the Lipschitz constant of φ_2 w.r.t. ξ_2 (of assumption 1).

This "Thau-like" condition [7] ensures that states of the following subsystem:

$$(\Sigma_{\xi_2}) \quad \dot{\xi}_2 = A_2\xi_2 + \varphi_2(\xi_2, v_1, u), \quad (v_1, u) \in \mathcal{AC}(I\!R^+, I\!R^{n_1}) \times \mathcal{U} \tag{3.10}$$

can be estimated by an "uncorrected" copy of its dynamics as follows:

$$(\mathcal{O}_{\xi_2}) \quad \dot{\hat{\xi}}_2 = A_2\hat{\xi}_2 + \varphi_2(\hat{\xi}_2, v_1, u). \tag{3.11}$$

Now we state the following:

Theorem 3.1 *If assumptions 1 and 2 hold, then the following system:*

$$(\mathcal{O}_\xi)\left\{\begin{array}{lcl} 0 & = & \theta S_\theta + A_1^T S_\theta + S_\theta A_1 - C^T C \\ \dot{\hat{\xi}}_1 & = & A_1\hat{\xi}_1 + \varphi_1(\hat{\xi}_1, \hat{\xi}_2, u) - S_\theta^{-1} C^T (C\hat{\xi}_1 - y) \\ \dot{\hat{\xi}}_2 & = & A_2\hat{\xi}_2 + \varphi_2(\hat{\xi}_2, \hat{\xi}_1, u) \end{array}\right. \tag{3.12}$$

is an asymptotic observer for system (3.6). Moreover, the rate of decay of the estimation error can be chosen to be as fast as the one imposed by sub-observer (3.11) of subsystem (3.10) (this is what we call "partial" correction).

Proof: The proof can be achieved by choosing appropriate candidate Lyapunov functions V_1 and V_2 in order to check assumption (H2) and apply result of proposition 2.1.

Take $\theta \geq 1$ and set $e_1 = \hat{\xi}_1 - \xi_1$ and $e_2 = \hat{\xi}_2 - \xi_2$. Then we get:

$$\left\{\begin{array}{lcl} \dot{e}_1 & = & (A_1 - S_\theta^{-1} C^T C)e_1 + \varphi_1(\hat{\xi}_1, \hat{\xi}_2, u) - \varphi_1(\xi_1, \xi_2, u) \\ \dot{e}_2 & = & A_2 e_2 + \varphi_2(\hat{\xi}_2, \hat{\xi}_1, u) - \varphi_2(\xi_2, \xi_1, u). \end{array}\right. \tag{3.13}$$

In the sequel, we will denote by e_{ij} the jth component of e_i, for $1 \leq j \leq n_i$ and $i = 1, 2$, and we set:

$$T(\theta) := \begin{pmatrix} \theta^{-1} & 0 & \dots & 0 \\ 0 & \theta^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \theta^{-n_1} \end{pmatrix}, \ \varepsilon_1 := T(\theta)e_1, \ \text{and} \ \varepsilon_2 := \frac{e_2}{\theta^{n_1}}. \tag{3.14}$$

A simple calculation shows that $T(\theta)S_1T(\Theta) = \dfrac{1}{\theta}S_\theta$ where $S_1 = S_\theta\mid_{\theta=1}$ (see e.g. [8]).

Now, take P, Q satisfying (3.9), and define V_1, V_2 by:

$$V_1(e_1) = \frac{1}{\theta} e_1^T S_\theta e_1 = e_1^T T(\theta) S_1 T(\theta) e_1 = \varepsilon_1^T S_1 \varepsilon_1 \tag{3.15}$$

$$V_2(e_2) = \frac{e_2^T P e_2}{\theta^{2n_1}} = \varepsilon_2^T P \varepsilon_2. \tag{3.16}$$

Obviously, V_1 and V_2 are positive definite, and to achieve the proof of theorem 3.1 we only need to show that V_1 and V_2 satisfy assumption (H2).

• Let us first check condition (H2)-(i) for V_1, V_2:

For V_1: here, the left-hand side of the inequality in (H2)-(i) becomes

$$\begin{array}{l} \dfrac{\partial V_1}{\partial e_1}[(A_1 - S_\theta^{-1}C^TC)e_1 + \varphi_1(\hat{\xi}_1, v_2, u) - \varphi_1(\xi_1, v_2, u)] \\ \quad = 2\varepsilon_1^T S_1 T(\theta)[(A_1 - S_\theta^{-1}C^TC)e_1 + \varphi_1(\hat{\xi}_1, v_2, u) - \varphi_1(\xi_1, v_2, u)]. \end{array} \tag{3.17}$$

On the one hand, using the triangular structure of φ_1 and the Lipschitz condition of assumption 1, we get:

$$\begin{aligned}
&\|T(\theta)[\varphi_1(\hat{\xi}_1, v_2, u) - \varphi_1(\xi_1, v_2, u)]\| \\
&\quad = (\sum_{i=1}^{n_1}[\frac{1}{\theta^i}\varphi_{1i}(\hat{\xi}_1, v_2, u) - \varphi_{1i}(\xi_1, v_2, u)]^2)^{\frac{1}{2}} \qquad (3.18)\\
&\quad \leq (\sum_{i=1}^{n_1}\frac{{\kappa_1}^2}{\theta^{2i}}\|\underline{e}_{1i}\|^2)^{\frac{1}{2}} \text{ (with } \underline{e}_{1i} = (e_{11}, \dots e_{1i})^T, \kappa_1 \text{ of assumpt.1)}\\
&\quad \leq n_1\kappa_1\|\varepsilon_1\| \quad \text{(from the definition of the } \varepsilon_{1i}\text{'s)},
\end{aligned}$$

and, on the other hand, a simple calculation gives:

$$T(\theta)(A_1 - S_\theta^{-1}C^TC)T(\theta)^{-1} = \theta(A_1 - S_1^{-1}C^TC). \qquad (3.19)$$

Hence using (3.18) and (3.19), we obtain:

$$\begin{aligned}
&\frac{\partial V_1}{\partial e_1}[(A_1 - S_\theta^{-1}C^TC)e_1 + \varphi_1(\hat{\xi}_1, v_2, u) - \varphi_2(\xi_1, v_2, u)]\\
&\quad = 2\varepsilon_1^T S_1\theta(A_1 - S_1^{-1}C^TC)\varepsilon_1 + 2\varepsilon_1^T S_1 T(\theta)(\varphi_1(\hat{\xi}_1, v_2, u) - \varphi_1(\xi_1, v_2, u))\\
&\qquad \text{(from (3.17) and (3.19))}\\
&\quad \leq \theta 2\varepsilon_1^T(S_1A_1 - C^TC)\varepsilon_1 + 2\kappa_1 n_1\lambda_{max}(S_1)\|\varepsilon_1\|^2 \quad \text{(using eq.(3.18))}\\
&\quad \leq -\theta V_1 - \theta\varepsilon_1^T C^TC\varepsilon_1 + 2\kappa_1 n_1\frac{\lambda_{max}(S_1)}{\lambda_{min}(S_1)}V_1\\
&\quad \text{(since } 2\varepsilon_1^T S_1A_1\varepsilon_1 = \varepsilon_1^T(S_1A_1 + A_1^TS_1)\varepsilon_1 = \varepsilon_1^T(-S_1 + C^TC)\varepsilon_1 \qquad (3.20)\\
&\qquad \text{and } \|\varepsilon_1\|^2 \leq \tfrac{1}{\lambda_{min}(S_1)}V_1)\\
&\quad \leq -(\theta - \mu_1)V_1, \quad \text{with } \mu_1 = 2\kappa_1 n_1\tfrac{\lambda_{max}(S_1)}{\lambda_{min}(S_1)} \text{ (independent of } \theta).
\end{aligned}$$

By taking $\theta > \mu_1$, it results that (H2)-(i) holds for V_1,
with $W_1 = (\theta - \mu_1)V_1$ (using notation W_1 of (H2)).

<u>For V_2:</u> in the same way, we have

$$\begin{aligned}
&\frac{\partial V_2}{\partial e_2}[A_2e_2 + \varphi_2(\hat{\xi}_2, v_1, u) - \varphi_2(\xi_2, v_1, u)]\\
&\quad = \frac{2}{\theta^{2n_1}}e_2^T P[A_2e_2 + \varphi_2(\hat{\xi}_2, v_1, u) - \varphi_2(\xi_2, v_1, u)] \qquad (3.21)\\
&\quad = -\varepsilon_2^T Q\varepsilon_2 + 2\varepsilon_2^T P\frac{1}{\theta^{n_1}}(\varphi_2(\hat{\xi}_2, v_1, u) - \varphi_2(\xi_2, v_1, u)) \text{ (see } P, Q, \varepsilon_2)\\
&\quad \leq -\lambda_{min}(Q)\|\varepsilon_2\|^2 + 2\bar{\kappa}_2\lambda_{max}(P)\|\varepsilon_2\|^2 \text{ (}\bar{\kappa}_2 \text{ as in assumption 2)}\\
&\quad \leq -\frac{(\lambda_{min}(Q) - 2\bar{\kappa}_2\lambda_{max}(P))}{\lambda_{max}(P)}V_2 := -\mu_2 V_2 \text{ with } \mu_2 > 0 \text{ (by (3.9))},
\end{aligned}$$

which gives condition (H2)-(i) for V_2, with $W_2 = \mu_2 V_2$.

• We then check condition (H2)-(ii) for V_1 and V_2:
For V_1: using now the Lipschitz condition w.r.t. ξ_2 and again the structure of φ_1, it results:

$$\begin{aligned}
&\|\frac{\partial V_1}{\partial e_1}[\varphi_1(\xi_1,\hat{\xi}_2,u)-\varphi_1(\xi_1,\xi_2,u)]\| \\
&\quad = \|2\varepsilon_1^T S_1 T(\theta)[\varphi_1(\xi_1,\hat{\xi}_2,u)-\varphi_1(\xi_1,\xi_2,u)]\| \qquad (3.22)\\
&\quad \leq \|2\varepsilon_1 S_1\| |\frac{1}{\theta^{n_1}}[\varphi_{1n_1}(\xi_1,\hat{\xi}_2,u)-\varphi_{1n_1}(\xi_1,\xi_2,u)]| \text{ (by structure of } \varphi_1)\\
&\quad \leq 2\lambda_{max}(S_1)\|\varepsilon_1\|\bar{\kappa}_1\|\varepsilon_2\| \quad (\bar{\kappa}_1 = \text{Lipschitz constant of } \varphi_1 \text{ w.r.t. } \xi_2)\\
&\quad \leq \frac{2\lambda_{max}(S_1)\bar{\kappa}_1}{\sqrt{\lambda_{min}(S_1)\lambda_{min}(P)}}\sqrt{V_1}\sqrt{V_2} \; (\|\varepsilon_1\|^2 \leq \tfrac{V_1}{\lambda_{min}(S_1)}, \|\varepsilon_2\|^2 \leq \tfrac{V_2}{\lambda_{min}(P)})\\
&\quad := \mu_{12}\sqrt{V_1}\sqrt{V_2}, \quad \text{with} \quad \mu_{12}>0 \quad (\text{independent of } \theta)
\end{aligned}$$

and thus $\|\frac{\partial V_1}{\partial e_1}[\varphi_1(\xi_1,\hat{\xi}_2,u)-\varphi_1(\xi_1,\xi_2,u)]\| \leq \frac{\mu_{12}}{\sqrt{(\theta-\mu_1)\mu_2}}\sqrt{W_1}\sqrt{W_2}$ i.e. condition (H2)-(ii) for V_1 is satisfied, with $\alpha_1 = \frac{\mu_{12}}{\sqrt{(\theta-\mu_1)\mu_2}}$.

For V_2: using the Lipschitz constant κ_2 of φ_2 w.r.t. ξ_1 (see assumption 1) we obtain:

$$\begin{aligned}
&\|\frac{\partial V_2}{\partial e_2}(\varphi_2(\xi_2,\hat{\xi}_1,u)-\varphi_2(\xi_2,\xi_1,u))\| \\
&\quad = \|\frac{2}{\theta^{n_1}}e_2^T P\frac{1}{\theta^{n_1}}(\varphi_2(\xi_2,\hat{\xi}_1,u)-\varphi_2(\xi_2,\xi_1,u))\|\\
&\quad \leq 2\lambda_{max}(P)\|\varepsilon_2\|\frac{\kappa_2}{\theta^{n_1}}\|e_1\| \qquad (3.23)\\
&\quad \leq 2\kappa_2\lambda_{max}(P)\|\varepsilon_2\|\|\varepsilon_1\| \text{ (using the definition of the } \varepsilon_{1i}\text{'s)}\\
&\quad \leq \frac{2\kappa_2\lambda_{max}(P)}{\sqrt{\lambda_{min}(S_1)\lambda_{min}(P)}}\sqrt{V_2}\sqrt{V_1}\\
&\quad := \mu_{21}\sqrt{V_2}\sqrt{V_1} \quad \text{with} \quad \mu_{21}>0 \quad (\text{independent of } \theta)
\end{aligned}$$

from which condition (H2)-(ii) follows for V_2, with $\alpha_2 = \frac{\mu_{21}}{\sqrt{(\theta-\mu_1)\mu_2}}$.

• Finally, choosing $\theta > \mu_1 + \frac{(\mu_{12}+\mu_{21})^2}{4\mu_2}$ gives $\alpha_1+\alpha_2 < 2$, i.e., (H2)-(iii), and by proposition 2.1, (3.12) is an asymptotic observer for (3.6). More precisely, with $V = V_1 + V_2$ and using results of (3.20)-(3.23) (where the

v_i's are specialized to $\hat{\xi}_i$), we obtain:

$$\begin{array}{c}\dot{V} \leq -(\theta-\mu_1)V_1 + (\mu_{12}+\mu_{21})\sqrt{V_1}\sqrt{V_2} - \mu_2 V_2, \\ \text{where } \mu_1, \mu_{12}, \mu_{21}, \mu_2 > 0 \text{ are independent of } \theta.\end{array} \tag{3.24}$$

Then, for any $0 < \eta < \mu_2$, choosing $\theta \geq \dfrac{(\mu_{12}+\mu_{21})^2}{4(\mu_2-\eta)} + \mu_1 + \eta$, ensures $\dot{V} \leq -\eta V$, i.e., exponential convergence, and the upper bound on η, i.e. the maximal rate of convergence of the observer, is clearly - by equation (3.21) - the one imposed by sub-observer (3.11) for subsystem (3.10). $\square$

Notice that if $n_1 = 1$, we are brought to a particular case of the form (3.2) with property (3.4), characterized in [1].

Remarking that the high-gain design of the form (3.8) can be obviously extended to multi-output systems (3.7) with $\xi_1 = \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1n_1} \end{pmatrix}$, $\xi_{1i} \in I\!R^\nu$, $y \in I\!R^\nu$, $A = \begin{pmatrix} 0 & Id_\nu & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & Id_\nu \\ 0 & \dots & 0 & 0 \end{pmatrix}$, $C = (Id_\nu \quad 0 \dots 0)$ and $\varphi_1(\xi_1, v_2, u) = \begin{pmatrix} \varphi_{11}(\xi_{11}, u) \\ \vdots \\ \varphi_{1n_1-1}(\xi_{11}, \dots \xi_{1n_1-1}, u) \\ \varphi_{1n_1}(\xi_1, v_2, u) \end{pmatrix}$, $\varphi_{1i} \in I\!R^\nu$, Id_ν denoting the $\nu \times \nu$ identity matrix, we can in the same way extend observer (3.12) to systems (3.6), where the subsystem (3.7) has several outputs and satisfy the above structure. This remark will be of particular interest in section 5.

4 Examples of Observers with Full Correction and Arbitrary Rate of Convergence

4.1 Cascade block-state affine systems

In the case of cascade structure (or weak interconnection), conditions for the existence of a transformation into a block state affine form (4.1) below, together with the existence of a corresponding observer are given in [5].

$$(\Sigma'_a)\left\{\begin{array}{lcl} \dot{x}_1 &=& A_1(u,y_1)x_1 + B_1(u,y_1), \quad x_1 \in I\!R^{n_1},\ x_2 \in I\!R^{n_2} \\ \dot{x}_2 &=& A_2(u,y_2,x_1)x_2 + B_2(u,y_2,x_1), \quad n_1+n_2=n \\ y &=& \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1x_1 \\ C_2x_2 \end{pmatrix}, \qquad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{array}\right. \tag{4.1}$$

where A_i and B_i are $\mathcal{C}^\infty$ functions w.r.t. their arguments.

Let us define the following subsystems:

$$(\Sigma'_{a1}) \qquad \dot{x}_1 = A_1(u, y_1)x_1 + B_1(u, y_1), \; y_1 = Cx_1 \tag{4.2}$$
$$(\Sigma'_{a2}) \qquad \dot{x}_2 = A_2(u, y_2, v_1)x_2 + B_2(u, y_2, v_1), \; y_2 = Cx_2 \tag{4.3}$$

and denote by $\chi(t, u, x^0)$ the solution of system (4.1) at time t starting at x^0, under control u, and π_1 the projection of $I\!R^n$ onto $I\!R^{n_1}$: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to x_1$.

Now consider, as in [5], the class of inputs u s.t.

$$(u, y_1), (u, y_2, \pi_1(\chi(t, u, x^0))$$

are "rich enough" (i.e., *regularly persistent* [5]) respectively for systems (4.2) and (4.3) and s.t. for any initial condition x in a compact set, $\chi(t, u, x)$ remains bounded. Then for any $\lambda > 0$, there exist θ_1, θ_2 such that the following system:

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & A_1(u, y_1)\hat{x}_1 + B_1(u, y_1) - S_{1\theta_1}^{-1} C_1^T (C_1 \hat{x}_1 - y_1) \\ \dot{S}_{1\theta_1} & = & -\theta_1 S_{1\theta_1} - A_1^T(u, y_1) S_{1\theta_1} - S_{1\theta_1} A_1(u, y_1) + C_1^T C_1 \\ \dot{\hat{x}}_2 & = & A_2(u, y_2, \hat{x}_1)\hat{x}_2 + B_2(u, y_2, \hat{x}_1) - \hat{S}_{2\theta_2}^{-1} C_2^T (C_2 \hat{x}_2 - y_2) \\ \dot{\hat{S}}_{2\theta_2} & = & -\theta_2 \hat{S}_{2\theta_2} - A_2^T(u, y_2, \hat{x}_1)\hat{S}_{2\theta_2} - \hat{S}_{2\theta_2} A_2(u, y_2, \hat{x}_1) + C_2^T C_2 \end{array} \tag{4.4}$$

is an observer for (4.1), satisfying $\|\hat{x} - x\| \leq \lambda_0 e^{-\lambda t}$. This is what we call *full* correction with *arbitrary rate of convergence*.

This result is proved in [5], but we can here check that conditions of proposition 2.2 are indeed satisfied: By considering regularly persistent inputs for (4.1), (H1') is satisfied since (4.2) admits the following observer for θ_1 large enough [2]:

$$(\mathcal{O}'_1) \left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & A_1(u, y_1)\hat{x}_1 + B_1(u, y_1) - S_{1\theta_1}^{-1} C_1^T (C_1 \hat{x}_1 - y_1) \\ \dot{S}_{1\theta_1} & = & -\theta_1 S_{1\theta_1} - A_1^T(u, y_1) S_{1\theta_1} - S_{1\theta_1} A_1(u, y_1) + C_1^T C_1 \end{array} \right. . \tag{4.5}$$

Here we are in a case where remark 2.3 applies and thus condition (H2') is satisfied. In the same way as above, (H3') also holds, with

$$(\mathcal{O}'_2) \begin{cases} \dot{\hat{x}}_2 & = & A_2(u, y_2, v_1)\hat{x}_2 + B_2(u, y_2, v_1) - S_{2\theta_2}^{-1} C_2^T (C_2 \hat{x}_2 - y_2) \\ \dot{S}_{2\theta_2} & = & -\theta_2 S_{2\theta_2} - A_2^T(u, y_2, v_1) S_1 - S_{2\theta_2} A_2(u, y_2, v_1) + C_2^T C_2 \end{cases} \tag{4.6}$$

and $v(t, e_2) := e_2^T S_{2\theta_2} e_2$.

Regular persistence of the inputs ensure that for θ_2 large enough, $\gamma_1 \|e_2\|^2 \leq v(t, e_2) \leq \gamma_2 \|e_2\|^2$, for some $\gamma_1, \gamma_2 > 0$ [5], and thus (H4') is also satisfied (and even its modified version of proposition 2.3).

4.2 Example of high gain observer design

If we now consider the case of two subsystems of the form (3.7) as considered in section 3, then the interconnection between them given hereafter can be subject to a high gain observer design with arbitrary convergence speed.

$$\begin{array}{rcl} \dot{x_1} & = & A_1 x_1 + \psi_1(x_1, x_2, u) \\ \dot{x_2} & = & A_2 x_2 + \psi_2(x_2, x_1, u) \\ y & = & \begin{pmatrix} C_1 x_1 \\ C_2 x_2 \end{pmatrix} \end{array} \tag{4.7}$$

with A_i, C_i in the form of A_1, C in (3.6) and $\psi_i(x_i, x_{\bar{\imath}}, u)^T = (\psi_{i1}(x_{i1}, u), \ldots, \psi_{i(n_i-1)}(x_{i1}, \ldots x_{i(n_i-1)}, u), \varphi_{in_i}(x_i, x_{\bar{\imath}}, u))$, $\{i, \bar{\imath}\} = \{1, 2\}$. ψ_1 and ψ_2 satisfy assumption 1 of section 3.

An observer is then given by

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & A_1\hat{x}_1 + \psi_1(\hat{x}_1, \hat{x}_2, u) - S_{1\theta_1}^{-1} C_1^T (C_1\hat{x}_1 - y_1) \\ \dot{\hat{x}}_2 & = & A_2\hat{x}_2 + \psi_2(\hat{x}_2, \hat{x}_1, u) - S_{2\theta_2}^{-1} C_2^T (C_2\hat{x}_2 - y_2) \end{array} \tag{4.8}$$

with $S_{1\theta_1}, S_{2\theta_2}$ given by

$$\begin{array}{rcl} \theta_1 S_{1\theta_1} + S_{1\theta_1} A_1 + A_1^T S_{1\theta_1} - C_1^T C_1 & = & 0 \\ \theta_2 S_{2\theta_2} + S_{2\theta_2} A_2 + A_2^T S_{2\theta_2} - C_2^T C_2 & = & 0 \end{array} \tag{4.9}$$

for some $\theta_1, \theta_2 > 0$, and $\forall \lambda > 0$, $\exists \theta_1, \theta_2$ s.t. $\|\hat{x} - x\| \leq \lambda_0 e^{-\lambda t}$.

This, obviously, is a particular case of multi-output *uniformly observable systems*, for which high-gain observers have been proposed in [8]. But here, we propose it as an illustrative example of conditions of proposition 2.1:

We first define subsystems (Σ_1) and (Σ_2) as follows:

$$\begin{array}{ll} (\Sigma_1) & \dot{x}_1 = A_1 x_1 + \psi_1(x_1, v_2, u), \;\; y_1 = C_1 x_1 \\ (\Sigma_2) & \dot{x}_2 = A_2 x_2 + \psi_2(x_2, v_1, u), \;\; y_2 = C_2 x_2. \end{array} \tag{4.10}$$

Then:

- Condition (H1) clearly holds with:

$$(\mathcal{O}_i) \left\{ \begin{array}{rcl} \dot{\hat{x}}_i & = & A_i\hat{x}_i + \psi_i(\hat{x}_i, v_{\bar{\imath}}, u) - S_{i\theta_i}^{-1} C_i^T (C_i\hat{x}_i - y_i) \\ 0 & = & \theta_i S_{i\theta_i} + S_{i\theta_i} A_i + A_i^T S_{i\theta_i} - C_i^T C_i \end{array} \right. \tag{4.11}$$

  and $V_i(e_i) = e_i^T T(\theta_i) S_{i\theta_i} T(\theta_i) e_i$ using notations (3.14).

- Easy computations then show that (H2)-(i) is satisfied with $W_i(e_i) = \theta_i V_i(e_i)$, that (H2)-(ii) follows from computations similar to those in the proof of theorem 3.1 (see equation (3.22)) and that (H2)-(iii) results from appropriate choice of θ_1, θ_2.

5 Illustrative Example

In order to put into relief how to use the approach proposed in this paper to design observers, let us consider the widely-studied example of an induction motor. Here is considered a classical two-phase equivalent Park representation of the motor, in a framework linked to the rotor, and usually denoted (α, β), with stator currents (I_s), stator fluxes (Φ_s) and rotor speed (Ω) as the state variables and stator voltages as the control variables. Since modeling is not the main purpose here, and since there are many references on induction motors (including full thesis such as [9], and very recent detailed papers such as [10]), we will not insist on that question. Just remark that denoting by Ω the speed of the motor, we only consider the case of a resistant torque of the form $\Gamma_{res} = K_0 + K_1\Omega$, with the constraint that K_0 and K_1 are assumed to be known.

In the sequel, X will denote the state, X_1 the vector of currents, X_2 the vector of fluxes and X_3 the speed.

5.1 Observer Design with Stator Currents Measurements

In a first approach, we will only consider stator currents as measured outputs, and show that the system then falls in the case of section 3.

From classical models, a representation w.r.t. our chosen state variables can indeed be derived using electrical and mechanical equations, as follows:

$$\begin{array}{rcl} \frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} & = & \left(\begin{array}{c|c} -aId_2 + p.\mathcal{H}X_3 & cId_2 - b.p.\mathcal{H}X_3 \\ \hline -R_sId_2 & 0 \end{array}\right)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + Bu \\ \dot{X}_3 & = & -\frac{K_1}{J}X_3 - \frac{K_0}{J} + \frac{p}{J}X_1^T\mathcal{H}X_2;\ B = (b.Id_2,\, Id_2)^T; \\ y & = & X_1 \in \mathbb{R}^2 \end{array} \tag{5.1}$$

where a, b, c, R_s, J, p are constant known parameters of the motor, Id_2 stands for the 2×2 identity matrix, and $\mathcal{H} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Denoting by h_1, h_2 the output functions, and f the drift of the system, we can define the following change of coordinates (where L_f stands for Lie derivative along f):

$$\begin{array}{lllllll} \xi_{11} & = & h_1(X), & \xi_{13} & = & L_f(h_1(X)) & \\ \xi_{12} & = & h_2(X), & \xi_{14} & = & L_f(h_2(X)), & \xi_2 = X_3. \end{array} \tag{5.2}$$

In these new coordinates, the system takes the following form

$$\begin{array}{rcl} \dot{\xi}_1 & = & A_1\xi_1 + \varphi_1(\xi_1, \xi_2, u) \\ \dot{\xi}_2 & = & A_2\xi_2 + \varphi_2(\xi_2, \xi_1) \\ y & = & C\xi_1 \end{array} \tag{5.3}$$

with $A_1 = \begin{pmatrix} 0 & Id_2 \\ 0 & 0 \end{pmatrix}$, $C = (I_2 \quad 0)$, $\varphi_1(\xi_1, \xi_2, u) = \begin{pmatrix} b.Id_2 u \\ \bar{\varphi}_1(\xi_1, \xi_2, u) \end{pmatrix}$ and $A_2 = -\dfrac{K_1}{J}$.

Using remarks at the end of section 3, and checking that assumptions of theorem 3.1 are satisfied, a high gain observer can be designed for the subsystem in ξ_1, and a complete observer is then derived as follows:

$$\begin{array}{rcl} 0 & = & -\theta S_\theta - A_1^T S_\theta - S_\theta A_1 + C^T C \\ \dot{\hat{\xi}}_1 & = & A_1\hat{\xi}_1 + \varphi_1(\hat{\xi}_1, \hat{\xi}_2, u) - S_\theta^{-1} C^T (C\hat{\xi}_1 - y) \\ \dot{\hat{\xi}}_2 & = & A_2\hat{\xi}_2 + \varphi_2(\hat{\xi}_2, \hat{\xi}_1). \end{array} \tag{5.4}$$

Note that whatever θ is, the convergence rate is in this case bounded by the one imposed by ξ_2, that is $-\dfrac{K_1}{J}$ which is unfortunately in general slower than electrical time constants. Using the following numerical values (in appropriate units) - corresponding to a motor with nominal power of $37\, kWatts$,

$$\begin{array}{lll} a = 60.72, & b = 501.3, & c = 807.1 \\ R_s = 0.07, & J = 0.41 & p = 2 \\ K_0 = 149.7 & K_1 = 0.002 & \end{array} \tag{5.5}$$

simulation results are given on figure 1, for $\theta = 500$.

This result illustrates the method, but is not really satisfactory, because the "uncorrected" estimation is quite slow (here increasing θ does not improve convergence speed).

Now if we consider the same system, with one more measurement, corresponding to an additional state (the rotor position, e.g. as in [11]), the speed estimation can then be also corrected, with arbitrary rate of convergence, as shown in next subsection.

5.2 Observer design with rotor position as additional measurement

Considering the rotor angular position measured, the system then becomes

$$\begin{array}{rcl} \frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} & = & A(X_3)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + Bu \\ \dot{X}_3 & = & -\frac{K_1}{J}X_3 - \frac{K_0}{J} + \frac{p}{J}X_1^T \mathcal{H} X_2 \\ \dot{X}_4 & = & X_3 \\ y & = & (X_1^T, X_4)^T \end{array} \tag{5.6}$$

with $A(X_3)$ and B as before (eq. (5.1)), and X_4 denoting the additional state.
If we perform the same change of coordinates, as in previous subsection, with now $x_1 := \xi_1$, $x_{21} := X_4$ and $x_{22} := X_3$, the system takes the same form as (4.7), with $A_1, \psi_1 = \varphi_1$ as in previous subsection, and $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\psi_2(x_2, x_1, u) = \begin{pmatrix} 0 \\ \varphi_2(x_2, x_1) \end{pmatrix}$ (φ_2 as in previous subsection).
Under this form, a complete observer of the form (4.8) can be obtained from the synthesis of two high gain observers, one for each subsystem, with arbitrary speed of convergence.

Simulation results are given in figure 2 with $\theta_1 = 2000, \theta_2 = 1000$. Note that with these new state and output, the whole system falls into the multi-output uniformly observable case as first studied in [8].

Finally, notice that, with this new measurement, the observer can be extended to the estimation of the constant load K_0 since extending the model with $\dot{K}_0 = 0$ and taking $x_{23} := \dot{X}_3$ does not change the structure (4.7).

6 Conclusion

In this paper we have derived conditions for observer design based on separate subdesigns. The conditions provided cover cases of partial correction in state estimation, which is a generalization of reduced observer design for nonlinear systems of former study [1], and full correction, as in previously proposed designs (block state affine cascade systems or multi-output "high gain" observable systems). Such a method is in principle very rich, but maybe as difficult to apply in a very general case as observer-based control, as underlined in the introduction. Nevertheless, it can be a starting point for developments of new synthesis of observers for nonlinear systems.

References

[1] G. Besançon and H. Hammouri. On uniform observation of nonuniformly observable systems, *Syst. & Control Let.*, **29** (1996), 9–19.

[2] G. Bornard, N. Couenne, and F. Celle. Regularly persistent observer for bilinear systems, *in Proc. of the Colloque International en Automatique Non Linéaire, Nantes*, June 1988.

[3] J. Rudolph and M. Zeitz. A block triangular nonlinear observer normal form, *Systems & Control Letters*, **23** (1994), 1–8.

[4] G. Besançon and G. Bornard. A condition for cascade time-varying linearization, *in IFAC Proc., Nonlinear Control Systems Design Symposium, Tahoe City, CA,USA*, 1995, 684–689.

[5] G. Besançon, G. Bornard, and H.Hammouri. Observer synthesis for a class of nonlinear control systems, *European Journal of Control*, **2** (1996), 176–192.

[6] J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems - applications to bioreactors, *IEEE Trans. on Automatic Control*, **37** (1992), 875–880.

[7] F.E. Thau. Observing the state of nonlinear dynamic systems, *Int. Journal of Control*, **17** (1973), 471–479.

[8] G. Bornard and H. Hammouri. A high gain observer for a class of uniformly observable systems, in *Proc. 30th IEEE Conf. on Decision and Control, Brighton, England, USA*, December 1992, 1494–1496.

[9] T. Von Raumer. *Commande Adaptative Non Linéaire de Machine Asynchrone*, PhD thesis, Institut Nat. Polytech. de Grenoble, 1994.

[10] J. Chiasson. Nonlinear controllers for an induction motor, *Control Eng. Practice*, **4** (1996), 977–990.

[11] M. Bodson, J. Chiasson, and R. Novotnak. Nonlinear speed observer for high-performance induction motor control, *IEEE Trans. on Industrial Electronics*, **42** (1995), 337–343.

Laboratoire d'Automatique de Grenoble, ENSIEG - BP 46, 38402 Saint Martin d'Hères, France

Laboratoire d'Automatique et Génie des Procédés, Université Claude Bernard Lyon I, 69622 Villeurbanne, France

Communicated by Clyde Martin