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A Guaranteed Filtering Scheme for Hereditary Linear Systems*

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Abstract

The problem of minimax guaranteed filtering [1] is considered for general time-delayed linear systems with uncertain input and output disturbances belonging to a ball in a Hilbert space, information about initial states being absent. Such problems were regarded under joint constraints for initial states and disturbances in papers [2, 3] as well. The conditions for infinite-dimensional informational sets to be bounded are given. The integral and differential evolutionary equations for the parameters of mentioned sets are derived, and a finite-dimensional approximation scheme for the problem is developed.

Key words: minimax guaranteed filtering, informational sets, hereditary differential systems, approximation scheme

AMS Subject Classifications: 49B40, 93E10

1 Introduction

This paper is concerned with the state estimation of a hereditary linear system, when there are uncertain input and output disturbances belonging to a ball in a Hilbert space, whereas initial state of the system is completely unknown. The uncertain quantities are not modelled as random variables or stochastic processes. In this case the initial states and disturbances are only known to lie in the given convex compacts in appropriate vector spaces. The estimation problem consists in construction of an informational set containing all possible states of the system and determined by the system

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dynamics, the bounds on the given ball, and available observation. This approach, also known as the process of guaranteed estimation, leads to an observer in the form of an evolution equation with set-valued solutions. For linear systems this approach was taken for the first time in [4, 5, 1] and independently in [6, 7]. For systems with delays, the state estimation problem was studied in [2], when initial states and disturbances are jointly constrained by an ellipsoid in a Hilbert space.

Other approaches based on stochastics can be found in papers [8, 9, 10, 11]. It had been usually supposed up till now that either the system had a more simple form or the joint constraints for the initial states and disturbances are present. In paper [3] the conditions of boundedness for the informational sets are given in terms of the restoration operator continuity and the spectral observability of stationary systems without information about initial states. However, there was considered only a static problem. In this work, the initial states for non-stationary equations are assumed to be also unknown. The defining correlations for the infinite-dimensional informational sets are obtained; and the necessary and sufficient conditions for them to be bounded are given. These are analogous to the conditions presented in paper [12]. The evolutionary equations are derived for the parameters of the informational sets and their approximations. Finally, the finite-dimensional approximation scheme for the problem is discussed. In section 3, for later use, we derive the defining correlations describing the structure of informational sets. It is done for an abstract problem in a Hilbert space. The obtained formulas are used further in our specific case for the derivation of evolutionary equations in section 4. Section 5 is devoted to some technical and numerical aspects of the problem.

2 Problem Formulation

Let a linear system with delays be described by the equations:

$$\dot{x} = \mathrm{A}(t)x_t + B(t)v(t), \quad y = \mathrm{G}(t)x_t + C(t)v(t), \quad t \in [t_0, t_1], \tag{2.1}$$

where $x \in R^n$, $y \in R^m$ is a measurable output vector. The uncertain vector function $v(t)$ is constrained by the inequality

$$\|v(\cdot)\|_{L_2^q(t_0,t_1)} \leq \nu. \tag{2.2}$$

The matrices $B(t)$, $C(t)$ are supposed to be piecewise continuous with rank $C(t) = m$ for each t. The symbol x_t is used in formulas (2.1) as the functional element $x_t(\theta) = x(t+\theta)$, $\theta \in [-h, 0]$, belonging to the Hilbert space

$$\mathcal{H} = R^n \times L_2^n(-h, 0). \tag{2.3}$$

The linear operator A(t) is defined for continuous functions on [-h,0] by the formula

$$\mathrm{A}(t)x_t = \sum_{i=0}^{N} A_i(t)x_t(-h_i) + \int_{-h}^{0} A(t,\theta)x_t(\theta)d\theta, \quad 0 = h_0 < \ldots < h_N = h, \tag{2.4}$$

with the matrices $A_i(\cdot)$, $A(\cdot,\cdot)$ being piecewise continuous. The linear operator G(t) has the same form as in (2.4) with symbol G instead of the A.

One has to describe the evolution of the infinite-dimensional informational set $\mathcal{X}_t = \mathcal{X}(t, y(\cdot))$.

Definition 2.1 *The informational set $\mathcal{X}_t = \mathcal{X}(t, y(\cdot))$ of states $x_t(\cdot)$ of system (2.1) that are consistent with measurement data $y(\cdot) = \{y(\theta),\, t_0 \leq \theta \leq t\}$ and with inequality (2.2) is the collection [1, 12] of all those elements $x_t(\cdot) \in \mathcal{H}$ for each of which there exists a function $v(\cdot) \in L_2^q(t_0, t)$ satisfying (2.2) and an initial state $x_{t_0}(\cdot) \in \mathcal{H}$ that generate the above element $x_t(\cdot)$ and the given signal $y(\cdot)$ in accordance with equations (2.1).*

Symbol Γ will be used further for the designation of the projection from $\mathcal{H}$ onto R^n. So $X_t = \Gamma\mathcal{X}_t$ is a finite-dimensional informational set considered in paper [2]. It is well known [13] that the solution of the first equation (2.1) may be represented as

$$x(t) = U(t,t_0)\phi_{t_0}(0) + \sum_{i=1}^{N}\int_{t_0}^{t_0+h_i} U(t,\theta)A_i(\theta)\phi(\theta - h_i)d\theta$$
$$+ \int_{-h}^{0}\int_{t_0}^{t_0-\theta} U(t,\tau)A(\tau,\theta)\phi(\theta+\tau)d\tau d\theta + \int_{t_0}^{t} U(t,\theta)B(\theta)v(\theta)d\theta, \tag{2.5}$$

where the fundamental matrix $U(t,\theta)$ is determined by the equation

$$\partial U(t,\theta)/\partial t = \mathrm{A}(t)U_t(\cdot,\theta), \quad t \geq \theta, \qquad U(t,t) = I, \quad U(t,\theta) = 0, \quad t < \theta. \tag{2.6}$$

Equation (2.5) uniquely defines [14, 10] the strongly continuous semigroup $\mathcal{U}(t,\tau)$, $t \geq \tau$, in space $\mathcal{H}$ with properties:

$$\mathcal{U}(t,\tau)\mathcal{U}(\tau,\alpha) = \mathcal{U}(t,\alpha),\ \alpha \leq \tau \leq t; \qquad \mathcal{U}(t,t) = id,$$
$$\partial\mathcal{U}(t,\tau)x/\partial t = \mathcal{A}_t\mathcal{U}(t,\tau)x, \tag{2.7}$$

where $\mathcal{A}_t$ is a linear unbounded operator determined by the equality $\mathcal{A}_t x = (\mathrm{A}(t)x, \dot{x}(\cdot))$ with everywhere dense in $\mathcal{H}$ domain $\mathcal{W}$ of definition. The set $\mathcal{W}$ consists of all absolutely continuous n-vector functions whose the derivatives belong to $L_2^n(-h,0)$. We note that the two-parameter semigroup operator $\mathcal{U}(t,\tau)$ satisfies the differential equation in (2.7) not only for $x \in \mathcal{W}$, but also for all $x \in \mathcal{H}$ if $t-\tau \geq h$. In addition, from (2.5) it turns out that operator $\mathcal{U}(t,\tau)$ under the condition $t-\tau \geq h$ being the Hilbert-Schmidt one has the integral representation

$$\mathcal{U}(t,\tau)x = (\int_{-h}^{0} U_1(t+\theta,\tau,\alpha)\phi(\alpha)d\alpha + U(t+\theta,\tau)l + \phi(t+\theta-\tau)$$

$$\times\chi(t+\theta-\tau,[-h,0)), \int_{-h}^{0} U_1(t,\tau,\alpha)\phi(\alpha)d\alpha + U(t,\tau)l),\ -h \leq \theta \leq 0. \quad (2.8)$$

Here $x=(l,\phi(\cdot))$; $U_1(t,\tau,\alpha)$ is a piecewise continuous function being equal to zero when $t=\tau$; the symbol $\chi(\cdot,A)$ means a characteristic function for set A. The semigroup operator $\mathcal{U}(t,\tau)$ enables us to write the formula

$$z(t) = \mathcal{U}(t,t_0)z_0 + \int_{t_0}^{t} \mathcal{U}(t,\tau)\Gamma^* B(\tau)v(\tau)d\tau, \quad (2.9)$$

which is equivalent in fact to the representation (2.5). The integral in (2.9) is taken in Bochner's sense. The formula has meaning for each $z_0 \in \mathcal{H}$, but only for $z_0 \in \mathcal{W}$ we have $z(t) \in \mathcal{W}$, $\forall t \geq t_0$, with the observation operator $\mathrm{G}(t)z(t) = \mathrm{G}(t)x_t$ being bounded. We may put

$$\mathcal{W} = \{(\phi_0,\phi(\cdot)) \in R^n \times H_1^n(-h,0) : \phi_0 = \phi(0)\},$$

where $H_1^n(-h,0)$ is a Hilbert space of functions with the inner product

$$[\phi,\psi]_{H_1^n} = \int_{-h}^{0} (\phi'(\theta)\psi(\theta) + \dot{\phi}'(\theta)\dot{\psi}(\theta))d\theta.$$

From now on the symbol $'$ means the transposition. Sometimes for technical reasons one requires more smooth initial states $z_0 \in R^n \times H_2^n(-h,0)$, where the Hilbert space $H_2^n(-h,0)$ has the inner product

$$[\phi,\psi]_{H_2^n} = \int_{-h}^{0} (\phi'(\theta)\psi(\theta) + \dot{\phi}'(\theta)\dot{\psi}(\theta) + \ddot{\phi}'(\theta)\ddot{\psi}(\theta))d\theta.$$

If so, then definition 2.1 have to be slightly modified.

3 The Defining Correlations for Linear Systems in a Hilbert Space

Let linear bounded operators $\mathcal{T}_i : \mathcal{B}_i \to \mathcal{H}$, $\mathcal{M}_i : \mathcal{B}_i \to \mathcal{B}$, i=0,1, be given, where $\mathcal{B}_i$, i=0,1, $\mathcal{B}$, $\mathcal{H}$ are real Hilbert spaces. Consider equations

$$x = \mathcal{T}_0\phi + \mathcal{T}_1 v, \qquad y = \mathcal{M}_0\phi + \mathcal{M}_1 v, \quad (3.1)$$

in which the element v is constrained by a priori restriction

$$\|v\|_{\mathcal{B}_1} \leq \nu, \tag{3.2}$$

where $\|\cdot\|_{\mathcal{B}_1}$ is a norm in space $\mathcal{B}_1$. Introduce some definitions.

Definition 3.1 *A pair (ϕ, v) is said to be consistent with the element y if the second equality in (3.1) and inequality (3.2) are fulfilled for this pair. A set of all consistent pairs will be denoted by Φ_y.*

Definition 3.2 *A set $\mathcal{X}_y \subset \mathcal{H}$ is called the informational one being consistent with the element y if it is the image of Φ_y according to the first equality in (3.1), i.e. $\mathcal{X}_y = \{x \in \mathcal{H} : x = \mathcal{T}_0\phi + \mathcal{T}_1 v, \quad (\phi, v) \in \Phi_y\}$.*

Given realized element y, the above sets are non-void, convex, and the set Φ_y is closed. For further constructions we accept the following

Assumption 3.3 *The operator $S = \mathcal{M}_1\mathcal{M}_1^*$ has the continuous inverse in space $\mathcal{B}$.*

Let us introduce a new inner product in space $\mathcal{B}$ as $[f, g]_1 = [f, S^{-1}g]$, where $[\cdot,\cdot]$ is the original product. The corresponding norm will be denoted by $\|\cdot\|_1$. The following assertion gives the structure of set Φ_y.

Lemma 3.4 *A pair $(\phi, v) \in \Phi_y$ iff*

$$v = v^0 + \tilde{v}, \quad \tilde{v} \in \ker\mathcal{M}_1, \quad v^0 = \mathcal{M}_1^* S^{-1}(y - \mathcal{M}_0\phi),$$
$$\|v\|^2 = \|v^0\|^2 + \|\tilde{v}\|^2 \leq \nu^2. \tag{3.3}$$

Proof: Note that $\|v^0\| = \|y - \mathcal{M}_0\phi\|_1 \leq \nu$. If conditions (3.3) hold then we have $\|v\| \leq \nu$ by virtue of $v^0 \perp \tilde{v}$. Since this fact take place, the second equality in (3.1) is obvious, and therefore $(\phi, v) \in \Phi_y$. Let the last inclusion be fulfilled. In view of correlations $\mathcal{M}_1 v^0 = y - \mathcal{M}_0\phi, \quad v - v^0 = \tilde{v} \in \ker\mathcal{M}_1, \quad v^0 \perp \tilde{v}$, we get conditions (3.3). Q.E.D.

Corollary 3.5 *An element $x \in \mathcal{X}_y$ iff*

$$x = \mathcal{T}_1\mathcal{M}_1^* S^{-1} y + (\mathcal{T}_0 - \mathcal{T}_1\mathcal{M}_1^* S^{-1}\mathcal{M}_0)\phi + \mathcal{T}_1\Pi_1 v, \; \|y - \mathcal{M}_0\phi\|_1^2 + \|v\|^2 \leq \nu^2, \tag{3.4}$$

where $\Pi_1 = id - \mathcal{M}_1^* S^{-1}\mathcal{M}_1$ is the orthogonal projection of $\mathcal{B}_1$ onto $\ker\mathcal{M}_1$.

Generally, the set $\mathcal{X}_y$ is not closed. However, its support function $\rho(f, \mathcal{X}_y) = \sup\{[f, x] : x \in \mathcal{X}_y\}$ characterizing a convex set with an accuracy of the closure [15] may be easily calculated if the following assumption takes place.

Assumption 3.6 *The element $f \in \mathcal{H}$ is an observable direction, i.e., $\mathcal{T}_0^* f \in im\mathcal{M}_0^*$, where symbol $im\mathcal{A}$ means the image of operator $\mathcal{A}$.*

It is known [10] that assumption 3.6 is equivalent to the inequality

$$\|\mathcal{M}_0\phi\| \geq \gamma|[f, \mathcal{T}_0\phi]| \qquad \forall\phi \in \mathcal{B}_0, \quad \gamma > 0. \tag{3.5}$$

The calculation of the support function is carried out sequentially. First one needs to maximize it in v, next in ϕ with regard to correlations (3.4). Denote by Π the orthogonal projection of $\mathcal{B}$ onto $\overline{\mathrm{im}\mathcal{M}_0}$ with respect to the inner product $[\cdot,\cdot]_1$. We have

$$\begin{aligned}\rho(f|\mathcal{X}_y) = \sup\{[f,x] : x \in \mathcal{X}_y\} = [f, \mathcal{T}_1\mathcal{M}_1^* S^{-1}y] + \sup_\phi\{[\lambda \\ -S^{-1}\mathcal{M}_1\mathcal{T}_1^* f, \mathcal{M}_0\phi] + (\nu^2 - \|y^\perp\|_1^2 - \|\Pi y - \mathcal{M}_0\phi\|_1^2)^{1/2}[f, Pf]^{1/2}\} \\ = [\Pi^*\lambda + (id - \Pi^*)S^{-1}\mathcal{M}_1\mathcal{T}_1^* f, y] + ([f, Pf] + \|\Pi(S\lambda \\ -\mathcal{M}_1\mathcal{T}_1^* f)\|_1^2)^{1/2}(\nu^2 - \|y^\perp\|_1^2)^{1/2},\end{aligned} \tag{3.6}$$

where

$$y^\perp = (id - \Pi)y, \quad P = \mathcal{T}_1\Pi_1\mathcal{T}_1^*, \quad \mathcal{M}_0^*\lambda = \mathcal{T}_0^* f. \tag{3.7}$$

When calculating we used the elementary equalities $\max_z\{[f,z] + (\nu^2 - \|z\|^2)^{1/2}a\}$=$\max_\alpha\{\|f\|\alpha + (\nu^2 - \alpha^2)^{1/2}a\} = \nu(\|f\|^2 + a^2)^{1/2}$. The maximum here is reached at $\alpha^0 = \nu\|f\|(\|f\|^2 + a^2)^{-1/2}$, $z^0 = \alpha^0 f\|f\|^{-1}$. So, the following assertion is true.

Theorem 3.7 *Under assumption 3.6 the support function of the set $\mathcal{X}_y$ is finite and given by formulas (3.6), (3.7).*

From this theorem it follows that if assumption 3.6 is fulfilled for each $f \in \mathcal{H}$, i.e. $\mathrm{im}\mathcal{T}_0^* \subset \mathrm{im}\mathcal{M}_0^*$, then the set $\mathcal{X}_y$ is bounded in the norm [16] for every element $y \in \mathcal{B}$. The condition $\mathrm{im}\mathcal{T}_0^* \subset \mathrm{im}\mathcal{M}_0^*$ may be called as one of the continuous observability (cf. [12]). For non-observable directions we have

Theorem 3.8 *Given a non-observable direction $f \in \mathcal{H}$, the support function $\rho(f|\mathcal{X}_y) < +\infty$ iff $\mathcal{T}_0^* f \perp ker\mathcal{M}_0$ and $\|y^\perp\|_1 = \nu$. In the latter case the set $\mathcal{X}_y = \{x\}$, i.e. it consists of a single point x.*

Proof: If $[\mathcal{T}_0^* f, \phi] > 0$ for some $\phi \in ker\mathcal{M}_0$ then it follows from (3.4) that $\rho(f|\mathcal{X}_y) = +\infty$. Suppose that $\mathcal{T}_0^* f \perp ker\mathcal{M}_0$ and the element $f \in \mathcal{H}$ is a non-observable direction. From (3.5) there exists such a sequence $\{\phi_n\}$ that $0 < n\|\mathcal{M}_0\phi_n\| < [f, \mathcal{T}_0\phi_n]$ $\forall n = 1, 2, \ldots$. Putting $\tilde{\phi}_n = \alpha\phi_n/\|\mathcal{M}_0\phi_n\|$, $q_n = \tilde{\phi} + \tilde{\phi}_n$, $\|\mathcal{M}_0\tilde{\phi}_n\| = \alpha > 0$ we will chose the element $\tilde{\phi}$ and the number α, so that $\|\Pi y - \mathcal{M}_0 q_n\|_1^2 < \nu^2 - \|y^\perp\|_1^2$ $\forall n$. We can do it if $\|y^\perp\|_1 < \nu$. By virtue of the inequality $[f, \mathcal{T}_0\tilde{\phi}_n] \geq \alpha n$ from (3.4), it immediately follows that $\rho(f|\mathcal{X}_y) = +\infty$. Finally, if $\|y^\perp\|_1 = \nu$, then $\mathcal{X}_y = \{x\}$, where x has the form as in (3.4) with $v = 0$, $\quad \mathcal{M}_0\phi = \Pi y$. Q.E.D.

For further consideration one needs to examine regular approximations of the set $\mathcal{X}_y$.

Definition 3.9 *A set $\mathcal{X}_y^{\epsilon k}$ is said to be the ϵk - informational one if it consists of all elements x given by formula (3.4) under the condition*

$$\epsilon\|\phi\|_{\mathcal{B}_0}^2 + \|y - \mathcal{M}_0\phi\|_1^2 + \|v\|_{\mathcal{B}_1}^2 \leq \nu^2 + k. \tag{3.8}$$

For every $k > 0$ the closed convex sets $\mathcal{X}_y^{\epsilon k} \neq \emptyset$ if the number ϵ is sufficiently small. We have inclusion $\mathcal{X}_y^{\epsilon_1 k} \subset \mathcal{X}_y^{\epsilon_2 k}$ if $\epsilon_2 < \epsilon_1$. Moreover

$$\cup_{\epsilon > 0} \mathcal{X}_y^{\epsilon k} = \mathcal{X}_y^{0k}, \tag{3.9}$$

where $\mathcal{X}_y^{0k}$ is the set given by (3.4), (3.8) with $\epsilon = 0$. If the condition 3.6 holds the support function of the latter set has the form (3.6), (3.7) with $\nu^2 + k$ instead of ν^2.

Let us find the support function of the closed set $\mathcal{X}_y^{\epsilon k}$. In just the same way as was done in formula (3.6), we get

$$\begin{aligned}\rho(f|\mathcal{X}_y^{\epsilon k}) = [f, \mathcal{T}_1\mathcal{M}_1^* S^{-1} y] + \sup_\phi \{[f, (\mathcal{T}_0 - \mathcal{T}_1\mathcal{M}_1^* S^{-1}\mathcal{M}_0)\phi] + (\nu^2 \\ + k - \|y - \mathcal{M}_0\phi\|_1^2 - \epsilon\|\phi\|^2)^{1/2}[f, Pf]^{1/2}\} = [f, Q_1^\epsilon y] + (\nu^2 + k \\ -[y, (id - \Pi^\epsilon)y]_1)^{1/2}[f, Q_2^\epsilon f]^{1/2},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}Q_1^\epsilon = \mathcal{T}_1\mathcal{M}_1^* S^{-1}(id - \Pi^\epsilon) + \mathcal{T}_0\overline{\mathcal{M}_\epsilon}\mathcal{M}_0^* S^{-1}, \qquad Q_2^\epsilon = P + \mathcal{K}^\epsilon, \\ \Pi^\epsilon = \mathcal{M}^\epsilon S^{-1},\ \mathcal{K}^\epsilon = (\mathcal{T}_0 - \mathcal{T}_1\mathcal{M}_1^* S^{-1}\mathcal{M}_0)\overline{\mathcal{M}_\epsilon}(\mathcal{T}_0^* - \mathcal{M}_0^* S^{-1}\mathcal{M}_1\mathcal{T}_1^*), \\ \mathcal{M}^\epsilon = \mathcal{M}_0\overline{\mathcal{M}_\epsilon}\mathcal{M}_0^*, \qquad \overline{\mathcal{M}_\epsilon} = (\mathcal{M}_0^* S^{-1}\mathcal{M}_0 + \epsilon id)^{-1}.\end{aligned} \tag{3.11}$$

The operator Q_2^ϵ is self-adjoint and non-negative defined. The following assertion is true.

Lemma 3.10 *The operators $\mathcal{M}^\epsilon$ and Π^ϵ from (3.11) strongly converge to $\mathcal{M}^0$ and $\Pi = \mathcal{M}^0 S^{-1}$ respectively as $\epsilon \to 0$.*

Proof: The operator Π^ϵ is self-adjoint as $[\Pi^\epsilon f, g]_1 = [\Pi^\epsilon f, S^{-1}g] = [f, \Pi^\epsilon g]_1$. The operator $\mathcal{M}^\epsilon$ satisfy the conditions: $\mathcal{M}^{\epsilon_1} \le \mathcal{M}^{\epsilon_2}$ if $\epsilon_2 \le \epsilon_1$,

$$\mathcal{M}^\epsilon = S - (S^{-1} + \epsilon^{-1} S^{-1} \mathcal{M}_0 \mathcal{M}_0^* S^{-1})^{-1} \le S.$$

Therefore [17], there exists the strong limit $\lim_{\epsilon \to 0} \mathcal{M}^\epsilon f = \mathcal{M}^0 f \quad \forall f$, where $\mathcal{M}^0$ is a bounded self-adjoint operator. Let us show that $\mathcal{M}^0 S^{-1} = \Pi$, i.e. the last operator represents the orthogonal projection from formula (3.6). If $f = \mathcal{M}_0 l$ then $[f, \Pi^\epsilon f]_1 = [f, f]_1 - \epsilon[l, l] + \epsilon^2 [l, \overline{\mathcal{M}}_\epsilon l]$. Passing to the limit in this equality we get that the self-adjoint with respect to $[\cdot,\cdot]_1$ operator $id - \mathcal{M}^0 S^{-1}$ vanishes on the subspace $\overline{\mathrm{im}\mathcal{M}_0}$. In addition, $\mathcal{M}^0 S^{-1} f = 0$ if $f \in ker(\mathcal{M}_0^* S^{-1}) = (im\mathcal{M}_0)^\perp$. Hence, it follows that $\mathcal{M}^0 S^{-1} = \Pi$. So, $\lim_{\epsilon \to 0} \Pi^\epsilon f = \Pi f \ \forall f$, with the convergence being regarded in the norm of space $\mathcal{B}$. Q.E.D.

From formulas (3.6), (3.7), (3.10), (3.11) and the Lemma 3.10, it immediately follows that

$$\lim_{\epsilon,\, k \to 0} \rho(f|\mathcal{X}_y^{\epsilon k}) = \rho(f|\mathcal{X}_y) \tag{3.12}$$

for the observable direction f. Moreover, if the system (3.1) is continuously observable, i.e., $\mathrm{im}\mathcal{T}_0^* \subset \mathrm{im}\mathcal{M}_0^*$, then we have the equivalent equality [10]

$$\mathcal{T}_0 = \Lambda \mathcal{M}_0 \tag{3.13}$$

for some linear bounded operator $\Lambda : \mathcal{B} \to \mathcal{H}$. Therefore, $\lambda = \Lambda^* f$ and the convergence in (3.12) is uniform in all f, $\|f\| = 1$. By virtue of the fact that the Hausdorff distance between convex bounded sets can be expressed by the support functions [1] we obtain

Theorem 3.11 *For each element y and $k > 0$ the sets $\mathcal{X}_y^{\epsilon k} \to \mathcal{X}_y^{0k}$ as $\epsilon \to 0$ in Hausdorff metric if the latter set is bounded. Besides, $\mathcal{X}_y^{0k} \to \mathcal{X}_y$ as $k \to 0$ in the same metric.*

4 The Evolutionary Equations for the Parameters of the Support Function of the Infinite-Dimensional Informational Set

In this section, we return to the original problem for equations (2.1) and attach the operators of system (3.1) to the concrete form with regard to formulas (2.5) - (2.9). We have here $\mathcal{B}_0 = \mathcal{H}$, $\mathcal{B}_1 = L_2^q(t_0, t)$, $\mathcal{B} = L_2^m(t_0, t)$, where $\mathcal{H}$ is Hilbert space (2.3). The space $\mathcal{W}$ will be endowed with H_1^n or

H_2^n-norm. The following equalities take place:

$$\mathcal{T}_0(t) = \mathcal{U}(t,t_0), \quad \mathcal{T}_1(t)v = \int_{t_0}^{t} \mathcal{U}(t,\tau)\Gamma^* B(\tau)v(\tau)d\tau,$$
$$\mathcal{M}_0(\cdot) = \mathrm{G}(\cdot)\mathcal{U}(\cdot,t_0), \quad \mathcal{M}_1(\cdot)v = \mathrm{G}(\cdot)\mathcal{T}_1(\cdot)v + C(\cdot)v(\cdot). \tag{4.1}$$

In these formulas the parameter t is added to the operators $\mathcal{T}_i,\ i=0,1$. The point in the designation of the operators $\mathcal{M}_i,\ i=0,1$, means that they are functions considered on the time interval $[t_0,t]$. Note that the symbol $\mathcal{M}_1^*(\alpha)\lambda$ now depends on α, and we have

$$\mathcal{M}_1^*(\alpha)\lambda = B'(\alpha)\int_{\alpha}^{t} U_G'(\tau,\alpha)\lambda(\tau)d\tau + C'(\alpha)\lambda(\alpha), \quad t_0 \le \alpha \le t,$$

where

$$U_G(t,\alpha) = \sum_{i=0}^{N} G_i(t)U(t-h_i,\alpha) + \int_{-h}^{0} G(t,\tau)U(t+\tau,\alpha)d\tau = \mathrm{G}(t)\mathcal{U}(t,\alpha)\Gamma^*.$$

Therefore, the equality $S(t)\lambda = y$ is the integral equation

$$S(t)\lambda(\tau) = L^{-1}(\tau)\lambda(\tau) + \int_{t_0}^{t} K(\tau,\theta)\lambda(\theta)d\theta = y(\tau), \quad t_0 \le \tau \le t, \tag{4.2}$$

where

$$L^{-1}(\tau) = C(\tau)C'(\tau), \quad K(\tau,\theta) = C(\tau)B'(\tau)U_G'(\theta,\tau)$$
$$+U_G(\tau,\theta)B(\theta)C'(\theta) + \int_{t_0}^{\tau\wedge\theta} U_G(\tau,\alpha)B(\alpha)B'(\alpha)U_G'(\theta,\alpha)d\alpha,$$
$$\tau\wedge\theta = \min\{\tau,\theta\}. \tag{4.3}$$

Here the integral kernel $K(\cdot,\cdot)$ is a piecewise continuous symmetric matrix function. For solving the equation (4.2) let us introduce the resolvent kernel $R(\tau,\alpha,t)$ as a solution to the matrix integral equations

$$R(\tau,\alpha,t) + \int_{t_0}^{t} K(\tau,\theta)L(\theta)R(\theta,\alpha,t)d\theta = K(\tau,\alpha) = R(\tau,\alpha,t)$$
$$+ \int_{t_0}^{t} R(\tau,\theta,t)L(\theta)K(\theta,\alpha)d\theta. \tag{4.4}$$

As it follows from Fredholm theory [17], the solution to the equation (4.4) being unique has the symmetric property: $R'(\tau,\alpha,t) = R(\alpha,\tau,t)$. So, with the help of the resolvent kernel we can write the solution to equation (4.2):

$$\lambda(\tau,t) = S^{-1}(t)y(\tau) = L(\tau)(y(\tau) - \int_{t_0}^{t} R(\tau,\theta,t)L(\theta)y(\theta)d\theta). \tag{4.5}$$

Let equality (4.2) define a family of strongly continuous in t operators in the space $L_2^m[t_0, t_1]$. Then from representation (4.4) we have that the function $R(\tau, \alpha, t)$ is piecewise continuous in $\tau, \alpha \leq t \leq t_1$. Moreover, it is smooth in t. To be more exact we establish

Lemma 4.1 *Let us assume that a vector function $y(\cdot)$ is continuous in a point $\bar{t}$ and besides so do the coefficients of equations (2.1) in this point. Then*

$$\partial S(t) y(\tau)/\partial t \mid_{t=\bar{t}} = K(\tau, \bar{t}) y(\bar{t})$$

and

$$\partial \lambda(\tau, t)/\partial t \mid_{t=\bar{t}} = -S^{-1}(\bar{t}) K(\tau, \bar{t}) \lambda(\bar{t}, \bar{t}),$$

where

$$\lambda(\tau, t) = S^{-1}(t) y(\tau), y(\cdot) \in L_2^m[t_0, t_1].$$

Proof: We get $S^{-1}(\bar{t}+\delta)y(\tau) - S^{-1}(\bar{t})y(\tau) = -S^{-1}(\bar{t}+\delta)[S(\bar{t}+\delta) - S(\bar{t})]S^{-1}(\bar{t})y(\tau)$ and, therefore, the operator $S^{-1}(t)$ is strongly continuous in the point $\bar{t}$. Note that the operator norms $\|S^{-1}(\bar{t}+\delta)\|$ in $L_2^m[t_0, t_1]$ are uniformly bounded in the neighborhood of the point $\bar{t}$. So, using the equalities (4.2) - (4.5) and passing to the limit yields

$$\lim_{\delta \to 0} (\lambda(\tau, \bar{t}+\delta) - \lambda(\tau, \bar{t}))/\delta = \lim_{\delta \to 0} \{-S^{-1}(\bar{t}+\delta)[S(\bar{t}+\delta) \\ -S(\bar{t})]S^{-1}(\bar{t})y(\tau)/\delta\} = -S^{-1}(\bar{t})K(\tau, \bar{t})\lambda(\bar{t}, \bar{t}).$$

Q.E.D.

We can assert that the differential equalities of Lemma 4.1 hold for almost all $t \in [t_0, t_1]$

Corollary 4.2 *The resolvent kernel $R(\tau, \alpha, t)$ satisfies the conditions*

$$\partial R(\tau, \alpha, t)/\partial t = -R(\tau, t, t) L(t) R(t, \alpha, t) \tag{4.6}$$

for every given τ, α, where function $R(\tau, \alpha)$ is continuous, and all $t \geq \tau \vee \alpha$ except for a finite number of points.

Proof: First note from (4.4), (4.5) that

$$R(\tau, \alpha, t) = L^{-1}(\tau) S^{-1}(t) K(\tau, \alpha).$$

As the matrix function $K(\tau, \alpha)$ is piecewise continuous, we can apply the Lemma 4.1 for $y(\tau) = K(\tau, \alpha)$ and $\lambda(\tau, t) = L(\tau) R(\tau, \alpha, t)$ with α being fixed. Q.E.D.

Consider function $R(t,\alpha,t)$ in more detail. Formulas (4.3), (4.4) yield

$$K(t,\alpha) = \mathrm{G}(t)\mathcal{U}_G(t,\alpha), \quad \mathcal{U}_G(t,\alpha) = \mathcal{U}(t,\alpha)\Gamma^* B(\alpha)C'(\alpha)$$

$$+\int_{t_0}^{\alpha} \mathcal{U}(t,\theta)\Gamma^* B(\theta)B'(\theta)U'_G(\alpha,\theta)d\theta, \; R(t,\alpha,t) = \mathrm{G}(t)F(t,\alpha)L^{-1}(\alpha),$$

$$F^*(t,\alpha) = S^{-1}(t)\mathcal{U}^*_G(t,\alpha), \quad F(t,\alpha)L^{-1}(\alpha) = \mathcal{U}_G(t,\alpha) - \mathcal{W}(t,\alpha),$$

$$\mathcal{W}(t,\alpha) = \int_{t_0}^{\alpha} \mathcal{U}_G(t,\theta)L(\theta)R(\theta,\alpha,t)d\theta. \tag{4.7}$$

Now we pass to the derivation of the evolutionary equations. The above notations and formulas (3.4), (3.7) produce the following equalities

$$\mathcal{M}_1(\cdot)\mathcal{T}_1^*(t) = \mathcal{U}^*_G(t,\cdot), \quad P(t) = \int_{t_0}^{t} (\mathcal{U}(t,\tau)\Gamma^* B(\tau)B'(\tau)\Gamma\mathcal{U}^*(t,\tau)$$

$$-\mathcal{U}_G(t,\tau)F^*(t,\tau))d\tau.$$

From Corollary 4.2 it follows that the term $\mathcal{W}(t,\alpha)$ is differentiable in t. Therefore, we get

$$\mathcal{W}(t,\alpha) = \mathcal{U}(t,\alpha)\mathcal{W}(\alpha,\alpha) + \int_{\alpha}^{t} \mathcal{U}(t,\theta)F(\theta,\theta)R(\theta,\alpha,\theta)d\theta,$$

$$F^*(\alpha,\alpha) = L(\alpha)\mathcal{G}P(\alpha),$$

where the designation $\mathcal{G}P(t) = C(t)B'(t)\Gamma + \mathrm{G}(t)P(t)$ is introduced. Using formulas (4.7), Fubini's theorem, and substituting $\mathcal{W}(t,\alpha)$ into the expression for P(t), we can transform it and come to the conclusion.

Lemma 4.3 *The operator $P(t)$ satisfies the integral equation*

$$P(t) = \int_{t_0}^{t} \mathcal{U}(t,\tau)(\Gamma^* B(\tau)B'(\tau)\Gamma - (\mathcal{G}P(\tau))^* L(\tau)\mathcal{G}P(\tau))\mathcal{U}^*(t,\tau)d\tau, \tag{4.8}$$

where the integral term

$$\mathcal{P}(t) = \int_{t_0}^{t} \mathcal{U}(t,\tau)(\mathcal{G}P(\tau))^* L(\tau)\mathcal{G}P(\tau)\mathcal{U}^*(t,\tau)d\tau \tag{4.9}$$

is uniquely defined by the bilinear form

$$[f,\mathcal{P}(t)g]_{\mathcal{H}} = \int_{t_0}^{t} (\mathcal{G}P(\tau)\mathcal{U}^*(t,\tau)f)' L(\tau)\mathcal{G}P(\tau)\mathcal{U}^*(t,\tau)g d\tau. \tag{4.10}$$

We have $P(t)f \in \mathcal{W}\ \forall f \in \mathcal{H}$, and $\mathcal{G}P(t)$ is an integral operator. Therefore, the conjugate operator $(\mathcal{G}P(t))^*$ and the term (4.9) are correctly defined for almost all $t \geq t_0$. Besides, $P(t)$ is strongly continuous in t, and the inequality

$$\|\mathcal{G}P(\cdot)\mathcal{U}^*(t,\cdot)f\|_{L_2^m(t_0,t)} \leq k\|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H} \tag{4.11}$$

takes place uniformly in $t \in [t_0, t_1]$. The solution to equation (4.8) is unique in a class of operators with the named properties.

Note that the uniqueness of the solution is proved by using of Gronwall's lemma as it was done in the work [10]. Moreover, from work [11] it follows that $P(t) \in \mathcal{N}_S^+$, where $\mathcal{N}_S^+$ is a cone of self-adjoint and non-negative Hilbert-Schmidt operators in $\mathcal{L}(\mathcal{H})$. The equation (4.8) may be called by the integral Riccati one with zero initial conditions. It can be easily transformed in a differential form. We also have to be careful when the terms with $\mathrm{G}(t)$ are conjugated because of the unboundedness of the $\mathrm{G}(t)$ on space $\mathcal{H}$.

Now consider operators (3.11) for the approximating sets. We introduce the operator

$$M(t,\alpha) = S^{-1}(t)\mathcal{M}_0(\alpha) = L(\alpha)(\mathrm{G}(\alpha)\mathcal{U}(\alpha,t_0) - \int_{t_0}^{t} R(\alpha,\theta,t)L(\theta)\mathrm{G}(\theta)\mathcal{U}(\theta,t_0)d\theta). \tag{4.12}$$

Differentiating with respect to t yields:

$$\partial M(t,\alpha)/\partial t = -L(\alpha)R(\alpha,t,t)L(t)\mathrm{G}(t)\mathcal{E}(t),$$
$$\mathcal{E}(t) = \mathcal{U}(t,t_0) - \mathcal{V}(t), \qquad \mathcal{V}(t) = \int_{t_0}^{t} F(t,\theta)\mathrm{G}(\theta)\mathcal{U}(\theta,t_0)d\theta. \tag{4.13}$$

If we differentiate the operator $\mathcal{V}(t)$ using the first equality (4.13), then the following representation can be obtained

$$\mathcal{V}(t) = \int_{t_0}^{t} \mathcal{U}(t,\tau)(\mathcal{G}P(\tau))^* L(\tau)\mathrm{G}(\tau)\mathcal{E}(\tau)d\tau. \tag{4.14}$$

Doing the same with the operator $\mathcal{M}_0^*(\cdot)S^{-1}\mathcal{M}_0(\cdot)$, we get

$$\mathcal{M}_0^*(\cdot)S^{-1}\mathcal{M}_0(\cdot) = \int_{t_0}^{t} (\mathrm{G}(\tau)\mathcal{U}(\tau,t_0))^* M(t,\tau)d\tau = \int_{t_0}^{t} (\mathrm{G}(\tau)\mathcal{E}(\tau))^* L(\tau)\mathrm{G}(\tau)\mathcal{E}(\tau)d\tau.$$

Observe that $\mathcal{T}_0 - \mathcal{T}_1\mathcal{M}_1^* S^{-1}\mathcal{M}_0 = \mathcal{U}(t,t_0) - \int_{t_0}^{t} \mathcal{U}_G(t,\tau)M(t,\tau)d\tau = \mathcal{E}(t)$. Hence, from (3.11), (4.13), (4.14) we have

$$\mathcal{K}^\epsilon(t) = \mathcal{E}(t)\overline{\mathcal{M}_\epsilon}(t)\mathcal{E}^*(t), \quad \overline{\mathcal{M}_\epsilon}(t) = (\int_{t_0}^{t} (\mathrm{G}(\tau)\mathcal{E}(\tau))^* L(\tau)\mathrm{G}(\tau) \times\mathcal{E}(\tau)d\tau + \epsilon\, id)^{-1}. \tag{4.15}$$

Using the evolutionary representation

$$\mathcal{E}(t) = \mathcal{U}(t,\alpha)\mathcal{E}(\alpha) - \int_\alpha^t \mathcal{U}(t,\tau)F(\tau,\tau)\mathrm{G}(\tau)\mathcal{E}(\tau)d\tau$$

and differentiability of $\overline{\mathcal{M}_\epsilon}(t)$, yield

$$\mathcal{K}^\epsilon(t) = \mathcal{U}(t,t_0)\mathcal{U}^*(t,t_0)\epsilon^{-1} - \int_{t_0}^t \mathcal{U}(t,\tau)(F(\tau,\tau)\mathrm{G}(\tau)\mathcal{K}^\epsilon(\tau)$$
$$-(\mathrm{G}(\tau)\mathcal{K}^\epsilon(\tau))^* F^*(\tau,\tau) + L(\tau)\mathrm{G}(\tau)\mathcal{K}^\epsilon(\tau))\mathcal{U}^*(t,\tau)d\tau.$$

Adding this equality to (4.8) we get

$$Q_2^\epsilon(t) = \mathcal{U}(t,t_0)\mathcal{U}^*(t,t_0)\epsilon^{-1} + \int_{t_0}^t \mathcal{U}(t,\tau)(\Gamma^* B(\tau)B'(\tau)\Gamma$$
$$-(\mathcal{G}Q_2^\epsilon(\tau))^* L(\tau)\mathcal{G}Q_2^\epsilon(\tau))\mathcal{U}^*(t,\tau)d\tau. \qquad (4.16)$$

This integral equation has the same structure as the equation (4.8), but its initial state is $Q_2^\epsilon(t_0) = \epsilon^{-1}\, id$. The uniqueness of the solution also takes place, and it has property (4.11) as in Lemma 4.3. However, $Q_2^\epsilon(t)f \in \mathcal{W}\ \ \forall f \in \mathcal{H}$, and the operator $\mathcal{G}Q_2^\epsilon(t)$ is integral only if $t \geq t_0+h$. Therefore, equations (4.15), (4.16) should be understood in a weak sense on the initial interval $[t_0, t_0+h]$ as in (4.10). In what follows it is necessary to bear this fact in the mind.

Let us derive an equation for the value

$$\hat{z}^\epsilon(t) = Q_1^\epsilon(t)y = \mu(t) + \gamma^\epsilon(t). \qquad (4.17)$$

Here we have

$$\mu(t) = \mathcal{T}_1(t)\mathcal{M}_1^*(\cdot)S^{-1}y = \int_{t_0}^t \mathcal{U}_G(t,\tau)\lambda(\tau,t)d\tau,$$
$$\gamma^\epsilon(t) = \mathcal{E}(t)\overline{\mathcal{M}_\epsilon}(t)\mathcal{M}_0^*(\cdot)S^{-1}y.$$

Using Lemma 4.1 and taking into account formulas (4.5), (4.7), we get

$$\mu(t) = \int_{t_0}^t \mathcal{U}(t,\tau)(\mathcal{G}P(\tau))^* L(\tau)(y(\tau) - \mathrm{G}(\tau)\mu(\tau))d\tau. \qquad (4.18)$$

Note that

$$\mathcal{M}_0^*(\cdot)S^{-1}y = \int_{t_0}^t M^*(t,\tau)y(\tau)d\tau = \int_{t_0}^t M^*(\tau,\tau)(y(\tau) - \mathrm{G}(\tau)\mu(\tau))d\tau.$$

From this formula and the evolutionary representation for $\mathcal{E}(t)$, $\overline{\mathcal{M}_\epsilon}(t)$ we derive the following one:

$$\gamma^\epsilon(t) = \int_{t_0}^t \mathcal{U}(t,\tau)((G(\tau)\mathcal{K}^\epsilon(\tau))^* L(\tau)(y(\tau) - \mathrm{G}(\tau)\hat{z}^\epsilon(\tau)) \\ -F(\tau,\tau)\mathrm{G}(\tau)\gamma^\epsilon(\tau))d\tau.$$

Now adding the last formula with (4.18), we have

$$\hat{z}^\epsilon(t) = \int_{t_0}^t \mathcal{U}(t,\tau)(\mathcal{G}Q_2^\epsilon(\tau))^* L(\tau)(y(\tau) - \mathrm{G}(\tau)\hat{z}^\epsilon(\tau))d\tau. \tag{4.19}$$

The linear integral equations (4.18), (4.19) have the unique solution. The last one represents the center of the ellipsoidal set $\mathcal{X}_t^{\epsilon k}$ with the support function (3.10). It remains still to get an equation for the value

$$\kappa^\epsilon(t) = [y, S^{-1}(id - \Pi^\epsilon)y] = \int_{t_0}^t y'(\tau)\lambda(\tau,t)d\tau \\ - \int_{t_0}^t y'(\tau)M(t,\tau)d\tau\overline{\mathcal{M}_\epsilon}(t)\mathcal{M}_0^*(\cdot)S^{-1}y.$$

By analogy with the foregoing we have

$$\kappa^\epsilon(t) = \int_{t_0}^t (y(\tau) - \mathrm{G}(\tau)\hat{z}^\epsilon(\tau))' L(\tau)(y(\tau) - \mathrm{G}(\tau)\hat{z}^\epsilon(\tau))d\tau. \tag{4.20}$$

Now discuss the possibility of passing to the limit as $\epsilon \to 0$ in equations (4.16), (4.19), (4.20). Denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from one Banach space X to another Y, $\mathcal{L}(X,X)$ is shortened with $\mathcal{L}(X)$. We can prove the following

Theorem 4.4 *Let system (2.1) be continuously observable at moment $t_\delta > t_0 + h$ that means according to (3.13) the existence of the operator $\Lambda(t_\delta) \in \mathcal{L}(L_2^m(t_0,t_\delta),\mathcal{H})$ for which*

$$\mathcal{U}(t_\delta,t_0) = \Lambda(t_\delta)G(\cdot)\mathcal{U}(\cdot,t_0). \tag{4.21}$$

Then all the values $Q_2^\epsilon(t)$, $Q_1^\epsilon(t)$, and $\kappa^\epsilon(t)$ strongly converge for each $t \geq t_\delta$ to $Q_2(t)$, $Q_1(t)$, and $\kappa(t)$ respectively. Moreover, these satisfy the equations

$$Q_2(t) = \mathcal{U}(t,t_\delta)Q_2(t_\delta)\mathcal{U}^*(t,t_\delta) + \int_{t_\delta}^t \mathcal{U}(t,\tau)(\Gamma^* B(\tau)B'(\tau)\Gamma \\ -(\mathcal{G}Q_2(\tau))^* L(\tau)\mathcal{G}Q_2(\tau))\mathcal{U}^*(t,\tau)d\tau;$$

$$\hat{z}(t) = Q_1(t)y = \mathcal{U}(t,t_\delta)\hat{z}(t_\delta) + \int_{t_\delta}^t \mathcal{U}(t,\tau)(\mathcal{G}Q_2(\tau))^* L(\tau)(y(\tau)$$

$$-\mathrm{G}(\tau)\hat{z}(\tau))d\tau;$$

$$\kappa(t) = \kappa(t_\delta) + \int_{t_\delta}^{t} (y(\tau) - \mathrm{G}(\tau)\hat{z}(\tau))' L(\tau)(y(\tau) - \mathrm{G}(\tau)\hat{z}(\tau))d\tau. \qquad (4.22)$$

Proof: First note that system (2.1) will be continuously observable at each moment $t \geq t_\delta$ as the equality like (4.21) holds for the operator $\Lambda(t) = \mathcal{U}(t, t_\delta)\Lambda(t_\delta)$. Using Lemma 3.10 we get the following strong limits

$$\lim_{\epsilon \to 0} \mathcal{K}^\epsilon(t) = (\Lambda(t) - \mathcal{T}_1(t)\mathcal{M}_1^*(\cdot)S^{-1})\mathcal{M}^0(\cdot)(\Lambda(t) - \mathcal{T}_1(t)\mathcal{M}_1^*(\cdot)S^{-1})^*$$

$$= \mathcal{K}^0(t), \qquad \lim_{\epsilon \to 0} \gamma^\epsilon(t) = (\Lambda(t) - \mathcal{T}_1(t)\mathcal{M}_1^*(\cdot)S^{-1})\mathcal{M}^0(\cdot)S^{-1}y = \gamma^0(t).$$

Therefore, the corresponding operators $Q_2^\epsilon(t)$, $Q_1^\epsilon(t)$ and value $\kappa^\epsilon(t)$ also have the strong limits for every $t \geq t_\delta$. Moreover, slightly modifying the proof of Lemma 3.10 we can establish the uniform convergence of the continuous in t functions $\mathcal{K}^\epsilon(t)f$, $\gamma^\epsilon(t)$ with range in $\mathcal{H}$ on $[t_\delta, t_1]$ for arbitrary $t_1 > t_\delta$. In spite of the operator $\mathrm{G}(t)$ for given t is not continuous on $\mathcal{H}$, using the property $Q_2^\epsilon(t)f \in \mathcal{W}\ \forall f \in \mathcal{W}$, we can assert the convergence of the products $\mathrm{G}(\cdot)Q_2^\epsilon(\cdot) \in \mathcal{L}(\mathcal{H}, L_2^m(t_\delta, t_1))$, $\mathrm{G}(\cdot)\hat{z}^\epsilon(\cdot)$ in space $L_2^m(t_\delta, t_1)$. It is sufficient to prove that for the operator $\mathrm{G}(t)x_t = G_1(t)x_t(-h)$ with one time delay. Passing to the limit under the integral sign yields the necessary equations (4.22). Q.E.D.

Remark 4.5 Further we will suppose that moment t_δ in (4.22) is increased by h. It will give us the opportunity to consider $Q_2(t_\delta) \in \mathcal{N}_S^+$ and to write with the help of the integral representation (2.8) the following formula

$$Q_2(t)f = (\int_{-h}^{0} E_1'(t,\tau)\phi(\tau)d\tau + E_0(t)l, \int_{-h}^{0} E_2(t,\cdot,\tau)\phi(\tau)d\tau$$
$$+E_1(t,\cdot)l), \qquad t \geq t_\delta, \qquad (4.23)$$

where $f = (l, \phi(\cdot)) \in \mathcal{H}$. Here we have $E_0'(t) = E_0(t) \geq 0$; $E_2'(t,\theta,\tau) = E_2(t,\tau,\theta)$ is a non-negative symmetric Hilbert-Schmidt kernel. We use the fact that the composition of an integral operator and an arbitrary bounded one is again an integral operator [17].

Along with formula (4.23) one can get a representation for $\hat{z}(t) = (e(t), e_1(t,\cdot)) \in \mathcal{H}$, $t \geq t_\delta$, with $e_1(t,\cdot) \in H_1^n$ (see equations (2.7)) and $e(t)$ being absolutely continuous in t. The inner product $[g, Q_2(t)f]$ is also absolutely continuous in t. For the corresponding approximate expressions we will add symbol ϵ to the designations.

In the following section we deduce differential equations for E_i and e_i. However, the integral equations (4.22) are more convenient [19, 11] with a view to the finite-dimensional approximation.

To finish this section let us discuss condition (4.21) of continuous observability. This condition is hard enough to check. Nevertheless, for the systems with constant coefficients the following result from [3] takes place. The stationary system (2.1) is continuously observable for each sufficiently large $t > 3^n n^2 + 4n + 2 + t_0$, where n is the dimension of the system, iff it is spectral completely observable, i.e.,

$$rank \begin{pmatrix} \overline{G}(\lambda) \\ \Delta(\lambda) \end{pmatrix} = n \qquad \forall \lambda \in C,$$

$$\overline{G}(\lambda) = \sum_{i=0}^{N} G_i \exp(-\lambda h_i) + \int_{-h}^{0} G(\tau) \exp(\lambda \tau) d\tau,$$

$$\Delta(\lambda) = \lambda I - \sum_{i=0}^{N} A_i \exp(-\lambda h_i) - \int_{-h}^{0} A(\tau) \exp(\lambda \tau) d\tau.$$

Here C is the set of all complex numbers.

5 A Decomposition of the Defining Correlations and a Finite-Dimensional Approximation Scheme

In many papers on optimal control for retarded differential equations [8, 9, 10, 18] as well as on the estimation problems [2] for theirs the defining partial differential systems are deduced. In our case it can be easily done. To this end, we multiply the first equation (4.22) from the left by $g = (m, \psi(\cdot))$ and from the right by $f = (l, \phi(\cdot))$, where $g, f \in \mathcal{H}$. Since the term $Q_2(t_\delta)\mathcal{U}^*(t, t_\delta) f \in \mathcal{W}$ for all $f \in \mathcal{H}$, with the help of (2.7) one can differentiate the obtained scalar integral equation for almost all $t \geq t_\delta$. We get

$$d[g, Q_2(t) f]/dt = [g, \mathcal{A}_t Q_2(t) f] + [f, \mathcal{A}_t Q_2(t) g] + (\Gamma g)' B(t) B'(t) \Gamma f \\ -(\mathcal{G} Q_2(t) g)' L(t) \mathcal{G} Q_2(t) f.$$

Substituting here equality (4.23) for $Q_2(t)$ and varying the elements g, f, yield ($\phi(\cdot) = \psi(\cdot) = 0$)

$$\dot{E}_0(t) = \mathrm{A}(t) Q_2(t) \Gamma^* + (\mathrm{A}(t) Q_2(t) \Gamma^*)' + B(t) B'(t) - (C(t) B'(t) \\ + \mathrm{G}(t) Q_2(t) \Gamma^*)' L(t) (C(t) B'(t) + \mathrm{G}(t) Q_2(t) \Gamma^*), \tag{5.1}$$

where

$$\mathrm{A}(t) Q_2(t) \Gamma^* = \sum_{i=0}^{N} A_i(t) E_1(t, -h_i) + \int_{-h}^{0} A(t, \tau) E_1(t, \tau) d\tau. \tag{5.2}$$

The expression $\mathrm{G}(t) Q_2 \Gamma^*$ has the same form as (5.2) with symbol G instead of the A. Besides, we have ($\phi(\cdot) = 0$, $m = 0$)

$$\partial E_1(t, \theta)/\partial t = \partial E_1(t, \theta)/\partial \theta + (\mathrm{A}(t) Q_2(t) \Delta^*)'(\theta) \\ -(\mathrm{G}(t) Q_2(t) \Delta^*)'(\theta) L(t) (C(t) B'(t) + \mathrm{G}(t) Q_2(t) \Gamma^*), \tag{5.3}$$

and ($l = 0,\ m = 0$)

$$\partial E_2(t,\theta,\tau)/\partial t = \partial E_2(t,\theta,\tau)/\partial\theta + \partial E_2(t,\theta,\tau)/\partial\tau$$
$$-(\mathrm{G}(t)Q_2(t)\Delta^*)'(\theta)L(t)\mathrm{G}(t)Q_2(t)\Delta^*(\tau). \qquad (5.4)$$

Here operator Δ is a projection from $\mathcal{H}$ onto $L_2^n(-h,0)$. Therefore,

$$\mathrm{G}(t)Q_2(t)\Delta^*(\tau) = \sum_{i=0}^{N} G_i(t)E_2(t,-h_i,\tau) + \int_{-h}^{0} G(t,\theta)E_2(t,\theta,\tau)d\theta. \quad (5.5)$$

Equations (5.1), (5.3), (5.4) are to be solved with the initial conditions $E_2(t_\delta,\theta,\tau)$, $E_1(t_\delta,\theta)$, $E_0(t_\delta)$ and the boundary ones

$$E_1(t,0) = E_0(t), \quad E_2(t,\cdot,0) = E_1(t,\cdot). \qquad (5.6)$$

Since the partial differential Riccati system (5.1), (5.3), (5.4) arises from the first equation in (4.22), it has a unique and indefinitely extendable solution. Note that the ϵ-approximating operator $Q_2^\epsilon(t)$ satisfying the integral equation (4.16) can also be represented in the form like (4.23) when $t \geq t_0 + h$. So, for the $Q_2^\epsilon(t)$ we can get the partial differential system as for the $Q_2(t)$. But on the initial segment $[t_0, t_0+h]$ the operator $Q_2^\epsilon(t)$ is not integral and not Hilbert-Schmidt.

Let us deduce the differential equations for the estimate $\hat{z}(t) = (e(t), e_1(t,\cdot)) \in \mathcal{H}$. Doing as above, from (4.22) we have

$$\dot{e}(t) = \mathrm{A}(t)\hat{z}(t) + (C(t)B'(t) + \mathrm{G}(t)Q_2(t)\Gamma^*)'L(t)(y(t) - \mathrm{G}(t)\hat{z}(t)),$$
$$\partial e_1(t,\theta)/\partial t = \partial e_1(t,\theta)/\partial\theta + (\mathrm{G}(t)Q_2(t)\Delta^*)'(\theta)L(t)(y(t)$$
$$-\mathrm{G}(t)\hat{z}(t)), \quad t \geq t_\delta, \qquad (5.7)$$

where

$$\mathrm{A}(t)\hat{z}(t) = \sum_{i=0}^{N} A_i(t)e_1(t,-h_i) + \int_{-h}^{0} A(t,\theta)e_1(t,\theta)d\theta, \quad e_1(t,0) = e(t). \qquad (5.8)$$

Using the obtained decomposition we can get the parameters of the finite-dimensional set $X_t = \Gamma\mathcal{X}_t$. Under condition (4.21) the support function of X_t has the form

$$\rho(l \mid X_t) = l'e(t) + (\nu - \kappa(t))^{1/2}(l'E_0(t)l)^{1/2}, \quad l \in R^n. \qquad (5.9)$$

Here vector $e(t)$ and matrix $E_0(t)$ satisfy equations (5.1) and (5.7). For the set X_t to be bounded for all $y(\cdot)$, it is necessary and sufficient that

$$\int_{t_0}^{t} \Lambda(t,\theta)\mathrm{G}(\theta)\mathcal{U}(\theta,t_0)d\theta = \Gamma\mathcal{U}(t,t_0), \qquad (5.10)$$

where $\Lambda(t,\cdot)$ is a $n\times m$-matrix function with elements from $L_2(t_0,t)$. It is clear that condition (4.21) implies (5.10) as we can set $\Lambda(t,\cdot)=\Gamma\Lambda(t)$. But it is not known yet whether condition (5.10), when it is fulfilled for all $t\geq t_\delta$, we will also consider the equation for the optimal approximating estimate:

$$\begin{aligned}\hat{x}_d(t) &= \mathcal{U}_d(t)\hat{z}(0)+\int_0^t \mathcal{U}_d(t-\tau)(\mathcal{G}_dQ_{2d}(\tau))^*L\\ &\quad\times(y(\tau)-G_d\hat{x}_d(\tau))d\tau.\end{aligned}$$

This finite-dimensional integral equation may be examined as an approximation for the initial equation

$$\begin{aligned}\hat{z}(t)=\mathcal{U}(t)\hat{z}(0)+\textstyle\int_0^t \mathcal{U}(t-\tau)(\mathcal{G}Q_2(\tau))^*L\\ \times(y(\tau)-\mathrm{G}\hat{z}(\tau))d\tau.\end{aligned}\qquad(5.11)$$

Now the convergence result can be formulated as follows.

Theorem 5.2 *For each $y(\cdot)\in L_2^m(t_0,t_1)$ generated by the system (2.1) the sequence $\hat{x}_d(t)$ converges to the solution $\hat{z}(t)$ of (5.11) in the space $C(0,t_1;\mathcal{H})$ of all continuous functions with the range in $\mathcal{H}$.*

For a detailed proof of the theorem, see paper [11].

We also obtain the convergence of the value

$$\kappa_d(t)=\kappa(0)+\int_0^t (y(\tau)-G_d\hat{x}_d(\tau))'L(y(\tau)-G_d\hat{x}_d(\tau))d\tau$$

to the initial $\kappa(t)$ from the equation (4.22). Therefore, from above formulas, theorems 5.1, 5.2 and results of sections 3, 4 it immediately follows that

$$\begin{aligned}\rho(f|\mathcal{X}_{td})=[f,\hat{x}_d(t)]+(\nu^2-\kappa_d(t))^{1/2}\\ \times[f,Q_{2d}(t)f]^{1/2}\to\rho(f|\mathcal{X}_t)\end{aligned}$$

uniformly in f, $\|f\|\leq 1$, as $d\to\infty$. The approximating informational set $\mathcal{X}_{td}$ is calculated for the finite-dimensional system in $R^{n(d+1)}$

$$\dot{\tilde{x}}_d=A_d\tilde{x}_d+Bv,\quad y=\tilde{G}_d\tilde{x}_d+Cv,\quad t\geq 0,$$

with $x_d=J_d\tilde{x}_d\in\mathcal{H}_d$, $\mathcal{X}_{td}=J_d\tilde{\mathcal{X}}_{td}$.

By virtue of the fact that the Hausdorff distance between convex bounded sets may be expressed by the support functions [1], we have $\mathcal{X}_{td}\to\mathcal{X}_t$, $t\geq 0$, as $d\to\infty$ in Hausdorff metric, if system (2.1) is continuously observable in the sense (4.21).

As a numerical example we consider the equations

$$\ddot{x} = \alpha(v(t-h) - \dot{x}(t-h)), \quad y = \dot{x}(t) + \xi(t).$$

This system describes the traffic problem for two transport vehicles. The second example deals with the problem of damping for an oscillatory system with the help of delayed feedback. The equations are

$$\ddot{x} = -\omega^2 x - cy, \quad y = \dot{x}(t-h) + v(t).$$

Computer simulation results for these examples concerning the evolution of the approximating informational sets will be published elsewhere.

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