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Disturbance Rejection by Dynamic Output Feedback: A Structural Solution*

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Abstract

We propose new structural necessary and sufficient conditions for the solvability of the Disturbance Rejection problem by dynamic Output Feedback. They generalize the necessary or sufficient conditions previously given by Commault, Dion and Benahcene. The structures which appear in our condition are related to the zeros at infinity and the unstable invariant zeros. We use both geometric and algebraic tools but with a particular attention to transfer function formulation.

Key words: linear systems, disturbance rejection, output feedback, infinite zeros, finite invariant zeros

AMS Subject Classifications: 93B52, 93C05, 93C35, 93C45, 93C60

1 Introduction

The Disturbance Rejection problem via dynamic Output Feedback (DROF) has received a lot of contributions. The problems with internal stability and pole placement have been solved by Willems and Commault [16] and Imai and Akashi [7] within a geometric approach. Algebraic counterparts (using transfer function matrices, stable rational fractional or polynomial fractional approaches) have also been given by Pernebo [15], Özgüler and Eldem [14], Eldem and Özgüler [6]. Explicit relations between geometric and algebraic approaches have been further enhanced in Özgüler [13].

For the particular Disturbance Rejection problem via State Feedback (DRSF), structural necessary and sufficient conditions for the existence

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of solutions have been provided. They rely on some particular structures of the infinite and finite zeros (see Malabre and Martinez Garcia [9] and also, for more compact structural solutions, Martinez Garcia et al. [12]). One major advantage of a structural treatment is that it leads to new relations between geometric and algebraic approaches and brings more insight into the solvability requirements. As concerns DROF, the paper by Commault, Dion and Benahcene [5] is the only reference where explicit structural conditions have been derived. However some of their conditions are necessary, others are sufficient and, except for some particular situations, no necessary and sufficient condition is available yet. The aim of this paper is to propose new necessary and sufficient structural solutions to the DROF problem, with a natural extension to the DROF problem with internal stability.

The paper is organized as follows: Section 2 is devoted to the notation and previous results. The main Section 3 describes our new solvability conditions, while Section 4 relates them to previously established (partial) results and illustrates our contributions through a simple example borrowed from [5].

2 Notation and Background

We shall consider here linear time invariant systems (A, B, C, D, E) described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ z(t) = Ex(t) \\ y(t) = Cx(t) \end{cases} \tag{2.1}$$

where $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ is the control input, $h(t) \in \mathcal{H} \approx \mathbb{R}^q$ is the disturbance input, $z(t) \in \mathcal{Z} \approx \mathbb{R}^l$ is the output to be controlled and $y(t) \in \mathcal{Y} \approx \mathbb{R}^p$ is the measured output. The same notation is used for maps and their matrix representations in particular bases $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$, $D : \mathcal{H} \to \mathcal{X}$, and $E : \mathcal{X} \to \mathcal{Z}$. We shall denote $\mathcal{B}$ the image of B, $\mathcal{D}$ the image of D, $\mathcal{C}$ the kernel of C and $\mathcal{E}$ the kernel of E. The letter s will be used for the Laplace variable.

The DROF problem amounts to looking for the existence of a dynamic compensator, say $u(s) = K(s)y(s)$, in such a way that the closed-loop transfer function matrix from $h(s)$ to $z(s)$ be identically zero.

The overall transfer function matrix being split as:

$$\begin{bmatrix} E(sI - A)^{-1}B & E(sI - A)^{-1}D \\ C(sI - A)^{-1}B & C(sI - A)^{-1}D \end{bmatrix} = \begin{bmatrix} G(s) & H(s) \\ M(s) & N(s) \end{bmatrix}, \tag{2.2}$$

it is well known (see for instance [5]) that the DROF problem is solvable if and only if the following equation:

$$G(s)X(s)N(s) = H(s) \tag{2.3}$$

has a proper solution $X(s)$.

Let us now recall some minimal geometric information (see [17] and [2]). A subspace $\mathcal{V}$ of $\mathcal{X}$ is called a Controlled (or $(A,\mathcal{B})$)-Invariant Subspace if $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$. Given any subspace $\mathcal{L} \subset \mathcal{X}$, there exists a Supremal Controlled (or $(A,\mathcal{B})$)-Invariant Subspace contained in $\mathcal{L}$, noted as $\mathcal{V}^*_\mathcal{L}$, and given as the limit of the following famous non increasing algorithm:

$$\begin{cases} \mathcal{V}'_\mathcal{L} = \mathcal{X} \\ \mathcal{V}^{\rangle+\infty}_\mathcal{L} = \mathcal{L}\cap\mathcal{A}^{-\infty}(\mathcal{V}^{\rangle}_\mathcal{L} + \mathcal{B}) \end{cases} \tag{2.4}$$

A subspace $\mathcal{S}$ of $\mathcal{X}$ is called a Conditioned (or $(\mathcal{C},\mathcal{A})$)-Invariant Subspace with respect to $\mathcal{C}$, if $A(\mathcal{S}\cap\mathcal{C}) \subset \mathcal{S}$. Given any subspace $\mathcal{M} \subset \mathcal{X}$, there exists an Infimal $(\mathcal{C},\mathcal{A})$ -Invariant Subspace containing $\mathcal{M}$, noted as $\mathcal{S}'^*_\mathcal{M}$, and given as the limit of the following famous non decreasing algorithm:

$$\begin{cases} \mathcal{S}''_\mathcal{M} = \prime \\ \mathcal{S}'^{\rangle+\infty}_\mathcal{M} = \mathcal{A}(\mathcal{S}'^{\rangle}_\mathcal{M} \cap \mathcal{C}) + \mathcal{M} \end{cases} . \tag{2.5}$$

The Supremal Internally Stabilizable Controlled (or $(A,\mathcal{B})$)-Invariant Subspace contained in $\mathcal{L}$, noted $\mathcal{V}^{*\ \int\sqcup\dashv\lfloor}_\mathcal{L}$, and the Infimal Externally Stabilizable Conditioned (or $(\mathcal{C},\mathcal{A})$)-Invariant Subspace containing $\mathcal{M}$, noted $\mathcal{S}'^{*\ \int\sqcup\dashv\lfloor}_\mathcal{M}$, can also be defined (see for instance [2]). We shall say that (2.1) is stabilizable and/or detectable when (A,B) is a stabilizable pair and/or (C,A) a detectable pair.

The geometric solvability condition for DROF is (see [16]):

Theorem 1 *There exists a solution to the DROF problem if and only if:*

$$\mathcal{S}'^*_\mathcal{D} \subset \mathcal{V}^*_\mathcal{E}. \tag{2.6}$$

Assuming that (2.1) is stabilizable and detectable, there exists an internally stable solution to the DROF problem if and only if:

$$\mathcal{S}'^{*\int\sqcup\dashv\lfloor}_\mathcal{D} \subset \mathcal{V}^{*\int\sqcup\dashv\lfloor}_\mathcal{E}. \tag{2.7}$$

When $C := Identity$, i.e. for state feedback solutions (DRSF problem), the subspaces on the left hand sides of (2.6) and (2.7) are simply $\mathcal{D}$. In that particular case, structural equivalent conditions (without and with stability requirements) have been proposed. They have been established with geometric tools in [9], but in order to make the exposition shorter, we shall just recall here the algebraic counterpart.

Theorem 2 *[9] There exists a solution to the DRSF problem if and only if $s^{-1}G(s)$ and $[s^{-1}G(s)\ ,H(s)]$ have the same infinite zero structure. Assuming that (2.1) is stabilizable, there exists an internally stable solution to the DRSF problem if and only if $s^{-1}G(s)$ and $[s^{-1}G(s)\ ,H(s)]$ have the same infinite zero structure and the same unstable* invariant *zero structure.*

This result (without stability considerations) can be directly obtained by formulating the DRSF problem in the following algebraic way: the DRSF problem is solvable if and only if the following equation:

$$G(s)Y(s) = H(s) \tag{2.8}$$

has a strictly proper solution.

The structure at infinity and the structure of finite invariant zeros have been given various equivalent definitions (see for instance the survey paper [1]). Let us simply recall here that, for a given system e.g. $G(s) = E(sI-A)^{-1}B$, they correspond to the so-called infinite and finite elementary divisors of the associated Rosenbrock system matrix $\begin{bmatrix} sI-A & B \\ E & 0 \end{bmatrix}$. These definitions obviously also hold true for row or column subsystems, e.g. for each row of $G(s)$, say $G_i^r(s)$ (for the i-th row) and for each column of $N(s)$, say $N_i^c(s)$ (for the i-th column). In the case of a single input or a single output subsystem, the infinite zero structure contains only one integer, the order of the zero at infinity.

Let us also quickly recall the notion of column essential orders (see [4] for the dual notion of row essential order). Let us denote $\overline{N}_i(s)$ the matrix $N(s)$ without its ith column. The essential order of the ith column of $N(s)$, say n_{ie}^c, is equal to the difference between the sum of the infinite zero orders of $N(s)$ and the sum of the infinite zero orders of $\overline{N}_i(s)$.

To our best knowledge, the main contributions towards a structural study of DROF are:

Lemma 3 *[5]: Assume that $G(s)$ is full row rank and $N(s)$ is full column rank. DROF is solvable if $G(s)$ and $[G(s), s^{n_{1e}^c}H_1^c(s), ..., s^{n_{qe}^c}H_q^c(s)]$ have the same infinite zero structure, where n_{ie}^c denotes the ith column essential order of $N(s)$, and $H_i^c(s)$ the ith column of $H(s)$.*

Lemma 4 *[5]: Assume that $G(s)$ is full row rank and $N(s)$ is full column rank. DROF is solvable only if $G(s)$ and $[G(s), s^{n_1^c}H_1^c(s), ..., s^{n_q^c}H_q^c(s)]$ have the same infinite zero structure, where n_i^c denotes the infinite zero order of the ith column of $N(s)$, and $H_i^c(s)$ the ith column of $H(s)$.*

These two conditions turn out to be necessary and sufficient [5] when $N(s)$ is column proper at infinity, i.e., when the infinite structure of $N(s)$ equals the union of the infinite structures of its columns.

It has to be noted that other necessary conditions have been derived in [3]. However no explicit "if and only if" structural condition is available yet. Our objective here is to propose a structural necessary and sufficient condition for the solvability of DROF, without any restrictive a priori assumption. Moreover, we shall also be able to do that under the additional internal stability requirement.

3 Main Results

Our first obvious trick amounts to defining an "extended" disturbance matrix, say D', such that:

$$\mathcal{D}' := \mathcal{I}\Updownarrow\mathcal{D}' = \mathcal{S}'^{*}_{\mathcal{D}}. \tag{3.9}$$

With this notation, it is clear that (2.6) is equivalent to:

$$\mathcal{D}' \subset \mathcal{V}^{*}_{\mathcal{E}}. \tag{3.10}$$

Similarly, let

$$\mathcal{D}'' := \mathcal{I}\Updownarrow\mathcal{D}'' = \mathcal{S}'^{*\int\sqcup\dashv\lfloor}_{\mathcal{D}}. \tag{3.11}$$

It is clear that (2.7) is equivalent to

$$\mathcal{D}'' \subset \mathcal{V}^{*\int\sqcup\dashv\lfloor}_{\mathcal{E}}.$$

Thus we directly get, from Theorems 1 and 2 and without needing any proof, the following structural equivalent:

Theorem 5 *Let $\mathcal{D}' := \mathcal{I}\Updownarrow\mathcal{D}' = \mathcal{S}'^{*}_{\mathcal{D}}$ and let $H'(s) := E(sI-A)^{-1}D'$. Then the DROF problem is solvable if and only if $s^{-1}G(s)$ and $[s^{-1}G(s), H'(s)]$ have the same infinite zero structure.*

Theorem 6 *Let $\mathcal{D}'' := \mathrm{Im}\,\mathcal{D}'' := \mathcal{S}'^{*\int\sqcup\dashv\lfloor}_{\mathcal{D}}$ and let $H''(s) := E(sI-A)^{-1}D''$. Under the assumption of stabilizability and detectability, the DROF problem has an internally stable solution if and only if $s^{-1}G(s)$ and $[s^{-1}G(s), H''(s)]$ have the same infinite and unstable invariant zero structures.*

In view of equation (2.8), we can reformulate Theorem 5 as follows:

Corollary 7 *Let $\mathcal{D}' := \mathcal{I}\Updownarrow\mathcal{D}' = \mathcal{S}'^{*}_{\mathcal{D}}$ and let $H'(s) := E(sI-A)^{-1}D'$. Then the DROF problem is solvable if and only if the equation $G(s)Y(s) = H'(s)$ admits a strictly proper solution $Y(s)$.*

At this level of exposition, it is worth pointing out that this new "explicit" condition is more attractive than the following implicit one, extracted for instance from [6] or [14]. Starting from (2.3), we can left and right multiply by some ad-hoc biproper matrices (proper, invertible and with proper inverse), say $B_i(s)$, in such a way that:

$$B_1(s)G(s)B_2(s)[B_2^{-1}(s)X(s)B_3^{-1}(s)]B_3(s)N(s)B_4(s) = B_1(s)H(s)B_4(s)$$

with:

$$B_1(s)G(s)B_2(s) = S^{\infty}_{Smith-McMillan}\left(G(s)\right) = \begin{bmatrix} diag(s^{-n_i}) & 0 \\ O & 0 \end{bmatrix}$$

$$B_3(s)N(s)B_4(s) = S^{\infty}_{Smith-McMillan}\left(N(s)\right) = \begin{bmatrix} diag(s^{-n'_i}) & 0 \\ O & 0 \end{bmatrix}$$

and where $S^{\infty}_{Smith-McMillan}(.)$ stands for the Smith McMillan Form at infinity of the transfer function matrix (the integers appearing as the powers of s^{-1} on the main diagonal are the orders of the zeros at infinity of (.)). Then, the existence of a proper solution for (2.3) can be reduced to the fact that some blocks of $B_1(s)H(s)B_4(s)$ must be zero, i.e., $B_1(s)H(s)B_4(s) = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix}$ and that $diag(s^{n'_i})\Phi diag(s^{n_i})$ is proper.

Unfortunately, these conditions rely on the particular transformations $B_1(s)$ and $B_4(s)$. Our conditions require no particular transformation of the initial data. However, one major drawback of our new formulation is that the transfer function matrix $H'(s)$ is directly dependent on some "geometric" information, namely $\mathcal{S}'^*_{\mathcal{D}}$.

The following result will show how $H'(s)$ can be directly obtained from $H(s)$ and $N(s)$ without any geometric intermediary. Let us first consider the following algorithm:

$$\begin{cases} \mathcal{V}'^{\prime}_{\mathcal{C}} = \mathcal{X} \\ \mathcal{V}'^{\rangle-\infty}_{\mathcal{C}} = \mathcal{C} \cap \mathcal{A}^{-\infty}(\mathcal{V}'^{\rangle}_{\mathcal{C}} + \mathcal{D}) \end{cases} \tag{3.12}$$

which limit is $\mathcal{V}'^*_{\mathcal{C}}$, the supremal $(A, \mathcal{D})$-invariant subspace contained in $\mathcal{C}$. From the results about "Almost Controllability Subspaces" (see for instance [8]), each step of $\mathcal{S}'^{\rangle}_{\mathcal{D}}$ (see algorithm (2.5)) can be rewritten as:

$$\mathcal{S}'^{\rangle}_{\mathcal{D}} = \mathcal{D} + \mathcal{A}(\mathcal{D} \cap \mathcal{V}'^{\infty}_{\mathcal{C}}) + \mathcal{A}^{\in}(\mathcal{D} \cap \mathcal{V}'^{\in}_{\mathcal{C}}) + \cdots + \mathcal{A}^{\rangle-\infty}(\mathcal{D} \cap \mathcal{V}'^{\rangle-\infty}_{\mathcal{C}}). \tag{3.13}$$

Let us define L_i as follows:

$$\begin{array}{l} \mathrm{Im}(DL_1) = \mathcal{D} \cap \mathcal{V}'^{\infty}_{\mathcal{C}} \\ \mathrm{Im}(DL_1L_2) = \mathcal{D} \cap \mathcal{V}'^{\in}_{\mathcal{C}} \\ \vdots \\ \mathrm{Im}(DL_1L_2\cdots L_i) = \mathcal{D} \cap \mathcal{V}'^{\rangle}_{\mathcal{C}}. \end{array} \tag{3.14}$$

Note that:

$$\mathcal{V}_{\mathcal{C}}^{\prime\rangle} = \left[\begin{array}{cc} I_{n\times n} & 0 \end{array}\right] \mathcal{K}\rceil\nabla \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ CA & CD & 0 & \cdots & 0 \\ CA^2 & CAD & CD & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{i-1} & CA^{i-2}D & CA^{i-3}D & & CD \end{bmatrix}. \tag{3.15}$$

Then from (3.13) and (3.14):

$$\mathcal{D}' := \mathcal{I}\Updownarrow\mathcal{D}' = \mathcal{S}_{\mathcal{D}}^{\prime *} = \tag{3.16}$$

$$Im\left[D\vdots A(DL_1)\vdots A^2(DL_1L_2)\vdots\cdots\vdots A^p(DL_1L_2\cdots L_p)\right] \tag{3.17}$$

with $p \in \mathbb{N}$, the smallest positive integer such that

$$\mathcal{S}_{\mathcal{D}}^{\prime\surd^{-\infty}} = \mathcal{S}_{\mathcal{D}}^{\prime\surd}. \tag{3.18}$$

We can thus rewrite $H'(s)$, defined by $H'(s) := E(sI-A)^{-1}D'$ (see Theorem 5) as follows:

$$H'(s) = E(sI-A)^{-1}\left[D\vdots A(DL_1)\vdots A^2(DL_1L_2)\vdots\cdots\vdots A^p(DL_1L_2\cdots L_p)\right]. \tag{3.19}$$

We will use this expression to prove the following intermediary result:

Lemma 8 *The DROF problem is solvable if and only if $s^{-1}G(s)$ and*

$$\left[s^{-1}G(s)\vdots\left(H(s)\vdots sH(s)L_1\vdots s^2H(s)L_1L_2\vdots\cdots\vdots s^pH(s)L_1L_2\ldots L_p\right)\right]$$

have the same infinite zero structure, where p and L_i are defined in (3.18) and (3.14).

Proof.
only if:
Consider that the DROF problem is solvable, i.e., $\mathcal{S}_{\mathcal{D}}^{\prime *} \subset \mathcal{V}_{\mathcal{E}}^{*}$. From (3.19), we can rewrite $H'(s)$ as follow:

$$H'(s) = \left[E(sI-A)^{-1}D\vdots\mathcal{E}(\int\mathcal{I}-\mathcal{A})^{-\infty}\mathcal{A}(\mathcal{D}\mathcal{L}_\infty)\vdots\cdots\vdots\mathcal{E}(\int\mathcal{I}-\mathcal{A})^{-\infty}\mathcal{A}^{\surd}(\mathcal{D}\mathcal{L}_\infty\mathcal{L}_\in\cdots\mathcal{L}_{\surd})\right] \tag{3.20}$$

For the first term of this expression, we have:

$$\boxed{E(sI-A)^{-1}D = H(s)}.$$

Let us expand $H(s)$ into its Laurent power series:

$$H(s) = E(sI-A)^{-1}D = \frac{ED}{s} + \frac{EAD}{s^2} + \frac{EA^2D}{s^3} + \cdots \tag{3.21}$$

Since $\mathcal{S}'^*_{\mathcal{D}} \subset \mathcal{V}^*_{\mathcal{E}}$ obviously $ED = 0$ (remember that $\mathcal{D} \subset \mathcal{S}'^*_{\mathcal{D}}$ and $\mathcal{V}^*_{\mathcal{E}} \subset \mathcal{E}$) and then:

$$E(sI-A)^{-1}AD = sH(s)$$

and consequently:

$$\boxed{E(sI-A)^{-1}ADL_1 = sH(s)L_1}.$$

Let us now consider the third term $E(sI-A)^{-1}A^2(DL_1L_2)$:
Since $\operatorname{Im} DL_1 = \mathcal{D} \cap \mathcal{K}\rceil\nabla\mathcal{C}$ and $A(\mathcal{D} \cap \mathcal{K}\rceil\nabla\mathcal{C}) \subset \mathcal{S}'^*_{\mathcal{D}} \subset \mathcal{V}^*_{\mathcal{E}} \subset \mathcal{E}$ then $EADL_1 = 0$. Now, from (3.21):

$$\begin{aligned} H(s)L_1 &= E(sI-A)^{-1}DL_1 = \frac{EA^2DL_1}{s^3} + \frac{EA^3DL_1}{s^4} + \cdots \\ &= \frac{1}{s^2}\left(\frac{EA^2DL_1}{s} + \frac{EA^3DL_1}{s^2} + \cdots\right) \\ &= \frac{1}{s^2}E(sI-A)^{-1}A^2DL_1 \end{aligned}$$

then:

$$E(sI-A)^{-1}A^2DL_1 = s^2H(s)L_1$$

and consequently:

$$\boxed{E(sI-A)^{-1}A^2DL_1L_2 = s^2H(s)L_1L_2}.$$

A similar treatment is possible for each term, until:

$$\boxed{E(sI-A)^{-1}A^pDL_1L_2\ldots L_p = s^pH(s)L_1L_2\ldots L_p}. \tag{3.22}$$

Then

$$\begin{aligned} &H'(s) := E(sI-A)^{-1}D' = \\ &\left[H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2 \vdots \cdots \vdots s^pH(s)L_1L_2\ldots L_p\right] . \end{aligned} \tag{3.23}$$

Finally from Theorem 5 if the DROF problem is solvable then $s^{-1}G(s)$ and

$$[s^{-1}G(s), H'(s)] = \left[s^{-1}G(s), \left(H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2 \vdots \cdots \vdots s^pH(s)L_1L_2\dots L_p\right)\right]$$

have the same infinite zero structure.
if:

Let us now assume that:
Assumption A:

$$\boxed{\left[s^{-1}G(s)\right] \text{and} \left[s^{-1}G(s) \vdots H_p(s)\right] \ \textit{have the same infinite zero structure}}$$

where:

$$H_p(s) := \left[H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2 \vdots \cdots \vdots s^pH(s)L_1L_2\dots L_p\right].$$

Let

$$\bar{D}_1 := D.$$

It is obvious that a necessary condition for *Assumption A* to hold is that $s^{-1}G(s)$ and $\left[s^{-1}G(s), H(s)\right] = \left[s^{-1}G(s), E(sI - A)^{-1}\bar{D}_1\right]$ have the same infinite zero structure too. Then, from Theorems 2 and 1 (with $\bar{D}_1$ instead of D): $\operatorname{Im}\bar{D}_1 \subset \mathcal{V}^*_{\mathcal{E}} \subset \mathcal{E}$ and then $ED = 0$.

From this follows that:

$$\begin{aligned} & \boxed{H(s) := E(sI - A)^{-1}D} \\ = \ & \frac{ED}{s} + \frac{EAD}{s^2} + \frac{EA^2D}{s^3} + \cdots = 0 + \frac{EAD}{s^2} + \frac{EA^2D}{s^3} + \cdots \\ = \ & \frac{1}{s}\left(\frac{EAD}{s} + \frac{EA^2D}{s^2} + \cdots\right) = \frac{1}{s}E(sI - A)^{-1}AD \end{aligned}$$

thus:

$$sH(s)L_1 = E(sI - A)^{-1}ADL_1$$

and

$$\boxed{\left[H(s) \vdots sH(s)L_1\right] = E(sI - A)^{-1}\left[D \vdots ADL_1\right]}.$$

Let

$$\bar{D}_2 := \left[D \vdots ADL_1\right].$$

Our structural *Assumption A* also implies that $\left[s^{-1}G(s), E(sI-A)^{-1}\bar{D}_2\right]$ and $s^{-1}G(s)$ have the same infinite zero structure. Then, from Theorems 2 and 1: (with $\bar{D}_2$ instead of D'): $\operatorname{Im}\bar{D}_2 \subset \mathcal{V}_{\mathcal{E}}^* \subset \mathcal{E}$ and then $ED = 0$ and $E(ADL_1) = 0$.

Then

$$\begin{aligned} H(s)L_1 &= E(sI-A)^{-1}DL_1 = \frac{EDL_1}{s} + \frac{EADL_1}{s^2} + \frac{EA^2DL_1}{s^3} + \cdots \\ &= 0 + 0 + \frac{EA^2DL_1}{s^3} + \cdots = \frac{1}{s^2}\left(\frac{EA^2DL_1}{s} + \cdots\right) \\ &= \frac{1}{s^2}E(sI-A)^{-1}A^2DL_1 \end{aligned}$$

and thus:

$$s^2H(s)L_1L_2 = E(sI-A)^{-1}A^2DL_1L_2$$

from which follows

$$\boxed{\left[H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2\right] = E(sI-A)^{-1}\left[D \vdots ADL_1 \vdots A^2DL_1L_2\right]}.$$

A similar treatment is possible for all intermediary steps until:

$$\boxed{\begin{array}{l} \left[H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2 \vdots \cdots \vdots s^pH(s)L_1L_2\ldots L_p\right] = \\ E(sI-A)^{-1}\left[D \vdots ADL_1 \vdots A^2DL_1L_2 \vdots \cdots \vdots A^pDL_1\ldots L_p\right]. \end{array}}$$

Let

$$\bar{D}_p := \left[D \vdots ADL_1 \vdots \cdots \vdots A^pDL_1\ldots L_p\right].$$

Under *Assumption A*, $s^{-1}G(s)$ and $\left[s^{-1}G(s), E(sI-A)^{-1}\bar{D}_p\right]$must have the same infinite zero structure and since $\operatorname{Im}\bar{D}_p := \mathcal{S}_{\mathcal{D}}^{\prime *}$ (see equation (3.17)), then from Theorems 2 and 1: $\operatorname{Im}\bar{D}_p = \mathcal{S}_{\mathcal{D}}^{\prime *} \subset \mathcal{V}_{\mathcal{E}}^*$ and thus the DROF problem is solvable. ■

We will now conclude this section by showing how L_i can be found directly from the Markov parameters of $N(s)$, without requiring any geometric intermediary. From the definition of L_i, and using (3.15), we have:

$$\operatorname{Im}(DL_1) := \mathcal{D} \cap \mathcal{V}_{\mathcal{C}}^{\prime\infty} = \mathcal{D}\cap\mathcal{K}\rceil\nabla\mathcal{C} = \mathcal{D}\mathcal{K}\rceil\nabla\mathcal{C}\mathcal{D}$$

$$\operatorname{Im}(DL_1L_2) = \mathcal{D} \cap \mathcal{V}_{\mathcal{C}}^{\prime\in} = \mathcal{D}\cap\left[\begin{array}{cc} I_{n\times n} & 0 \end{array}\right]\mathcal{K}\rceil\nabla\left[\begin{array}{cc} C & 0 \\ CA & CD \end{array}\right]$$

$$= D \left[\begin{array}{cc} I_{q\times q} & 0 \end{array} \right] Ker \left[\begin{array}{cc} CD & 0 \\ CAD & CD \end{array} \right] \tag{3.24}$$

$$\vdots$$

$$\mathrm{Im}(DL_1L_2\cdots L_p) = \mathcal{D} \cap \mathcal{V}_{\mathcal{C}}^{'\checkmark}$$

$$= \mathcal{D}\cap \left[\begin{array}{cccc} I_{n\times n} & 0 & \cdots & 0 \end{array} \right] \mathcal{K}\rceil\nabla \left[\begin{array}{cccc} C & 0 & \cdots & 0 \\ CA & CD & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CA^{p-1} & CA^{p-2}D & \cdots & CD \end{array} \right]$$

$$= D \left[\begin{array}{cccc} I_{q\times q} & 0 & \cdots & 0 \end{array} \right] Ker \left[\begin{array}{cccc} CD & 0 & \cdots & 0 \\ CAD & CD & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CA^{p-1}D & CA^{p-2}D & \cdots & CD \end{array} \right]. \tag{3.25}$$

Let us denote N_i the i-th Markov parameter of $N(s)$, i.e.:

$$N(s) := C(sI-A)^{-1}D = \frac{CD}{s} + \frac{CAD}{s^2} + \frac{CA^2D}{s^3} + \cdots = \frac{N_1}{s} + \frac{N_2}{s^2} + \frac{N_3}{s^3} + \cdots \tag{3.26}$$

We can thus rewrite Lemma 8, without needing any additional proof, in the following way:

Theorem 9 *Let us define $G(s)$, $H(s)$ and $N(s)$ as in (2.2) and L_i such that*

$$\mathrm{Im}(L_1) = Ker N_1$$

$$\mathrm{Im}(L_1L_2) = \left[\begin{array}{cc} I_{q\times q} & 0 \end{array} \right] Ker \left[\begin{array}{cc} N_1 & 0 \\ N_2 & N_1 \end{array} \right]$$

$$\vdots$$

$$\mathrm{Im}(L_1L_2\cdots L_p) = \left[\begin{array}{cccc} I_{q\times q} & 0 & \cdots & 0 \end{array} \right] Ker \left[\begin{array}{cccc} N_1 & 0 & \cdots & 0 \\ N_2 & N_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ N_p & N_{p-1} & \cdots & N_1 \end{array} \right]$$

where the N_i's are the Markov parameters of $N(s)$ (see (3.26)). The DROF problem is solvable if and only if there exists some $p \leq n$ such that:

$$s^{-1}G(s) \ \ and \ \left[s^{-1}G(s) \vdots H_p(s) \right]$$

have the same infinite zero structure, where

$$H_p(s) := \left[H(s) \vdots sH(s)L_1 \vdots s^2H(s)L_1L_2 \vdots \cdots \vdots s^pH(s)L_1L_2\ldots L_p \right].$$

4 Connection with the Previous Structural Conditions

In this section we show how our structural solution to the DROF problem, i.e., Theorem 9, is equivalent to that of Commault, Dion and Benahcene [5] (Lemma 4) in the particular case when the transfer function matrix $N(s)$ is column proper at infinity. For that, let us recall that (see [4]):

Proposition 10 *The matrix $N(s)$ is column proper at infinity, iff:*

$$n_i^c = n_{ie}^c \ ; \ \forall i = 1, ..., q$$

where n_i^c (n_{ie}^c) denotes the infinite zero order (the essential order) of the i-th column of $N(s)$ (see Section 2).

From Proposition 10, directly follows that all the non zero columns of N_1 are independent. The kernel of N_1 is thus generated by some elementary vectors of the basis of $\mathcal{H}$. This means that after some column permutations we can rewrite N_1 in the following way:

$$N_1 = \left[\begin{array}{ccc} 0_1 \cdots 0_{r_1} & \vdots & \overline{N}_1 \end{array} \right]$$

with $\overline{N}_1$ monic. Thus:

$$\operatorname{Im} L_1 = \left[\begin{array}{c} I_{r_1} \\ 0 \end{array} \right].$$

From the definition of L_i's (see Theorem 9):

$$\operatorname{Im}(L_1 L_2) = \left[\begin{array}{cc} I_{q \times q} & 0 \end{array} \right] Ker \left[\begin{array}{cc} N_1 & 0 \\ N_2 & N_1 \end{array} \right]$$

$$= Ker N_1 \cap N_2^{-1}(\operatorname{Im} N_1).$$

Because of Proposition 10:

$$\operatorname{Im}(L_1 L_2) = Ker N_1 \cap Ker N_2$$

Indeed, all the non zero columns of N_2 among the first r_1 ones are independent from those of N_1 (essentiality). Then $\operatorname{Im} L_1 L_2$ is generated by basis vectors of $\mathcal{H}$ corresponding to the columns of $N(s)$ which infinite zero order is greater than 2.

We can proceed in the same way for each term and show that under the assumption that $N(s)$ is column proper at infinity:

$$\operatorname{Im}(L_1 L_2 \ldots L_p) = Ker N_1 \cap Ker N_2 \cap \ldots \cap Ker N_p.$$

The structural solvability condition of Theorem 9, namely:

$$\boxed{\left[s^{-1}G(s)\right]\text{and}\left[s^{-1}G(s)\vdots H_p(s)\right] \textit{ have the same infinite zero structure}} \tag{4.27}$$

is obviously equivalent to

$$\boxed{[G(s)]\text{and}\left[G(s)\vdots sH_p(s)\right] \quad \textit{have the same infinite zero structure}} \tag{4.28}$$

with:

$$H_p(s) := \left[H(s)\vdots sH(s)L_1\vdots s^2H(s)L_1L_2\vdots\cdots\vdots s^pH(s)L_1L_2\ldots L_p\right].$$

Now, under the assumption that $N(s)$ is column proper at infinity, since $\mathrm{Im}(L_1L_2\ \ldots L_p)$ is generated by elementary vectors of the basis, i.e., the products $H(s)L_1L_2\ldots L_i$ exactly correspond to column selections in $H(s)$, it is quite direct to show that (4.28) is equivalent to

$$\boxed{[G(s)]\text{and}\left[G(s)\vdots H(s)diag(s^{n_i^c})\right] \textit{ have the same infinite zero structure,}}$$

which is the condition of Lemma 4. Indeed, each column of $H(s)$ appears in $\left[G(s)\vdots H_p(s)\right]$ with coefficients $s, s^2, \ldots$ up to $s^{n_i^c}$ and it is quite clear that for each column of $H(s)$, say $H_i^c(s)$,

$$\boxed{\begin{array}{c}[G(s)] \textit{ and } \left[G(s)\vdots sH_1^c(s)\vdots s^2H_2^c(s)\vdots\cdots\vdots s^{n_i^c}H_i^c(s)\right] \\ \textit{have the same infinite zero structure}\end{array}}$$

if and only if

$$\boxed{[G(s)]\text{and}\left[G(s)\vdots s^{n_i^c}H_i^c(s)\right] \textit{ have the same infinite zero structure}}.$$

As a matter of fact, it may be shown rather easily (just using essentiality notion in a deeper way) that the structure at infinity of $\left[G(s)\vdots sH_p(s)\right]$ is lower and upper bounded by the structure at infinity of, respectively, $\left[G(s)\vdots H(s)diag(s^{n_i^c})\right]$ and $\left[G(s)\vdots H(s)diag(s^{n_{ie}^c})\right]$. This fully explains the links between our necessary and sufficient condition and those (only necessary or sufficient) of Commault, Dion and Benahcene [5].

Before concluding, let us illustrate our results on the example extracted from [5].

Example 1 *Let:*

$$G(s) = \begin{bmatrix} s^{-1} & 0 \\ s^{-1} & s^{-2} \end{bmatrix}; \ H(s) = \begin{bmatrix} \alpha s^{-3} & 0 \\ 0 & \beta s^{-3} \end{bmatrix}; \ N(s) = \begin{bmatrix} s^{-1} & s^{-1} \\ 0 & s^{-2} \end{bmatrix}.$$

Then:

$$s^{-1}G(s) = \begin{bmatrix} s^{-2} & 0 \\ s^{-2} & s^{-3} \end{bmatrix}.$$

The infinite zero structure of $s^{-1}G(s)$ is $\{2,3\}$.

$$N_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \ ; N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ ; L_1 L_2 = 0$$

$$\left[s^{-1}G(s) \vdots H(s) \vdots sH(s)L_1 \right] = \begin{bmatrix} s^{-2} & 0 & \alpha s^{-3} & 0 & \alpha s^{-2} \\ s^{-2} & s^{-3} & 0 & \beta s^{-3} & -\beta s^{-2} \end{bmatrix}.$$

The infinite zero structure of $\left[s^{-1}G(s) \vdots H(s) \vdots sH(s)L_1 \right]$ is $\{2,2\}$ if $\alpha \neq -\beta$ and $\{2,3\}$ if $\alpha = -\beta$. Then, for this example, it is possible to reject the disturbance by output feedback if and only if $\alpha = -\beta$ (see Theorem 9).

5 Concluding Remarks

We have proposed structural necessary and sufficient conditions for the solvability of the DROF problem. These conditions are similar to (though more intricate than) the ones previously obtained for the state feedback case. Theorem 5, for instance, amounts to comparing the structures at infinity of two "artificial" systems: the one of the "undisturbed" system and that of a compound system where some "extended" disturbance inputs are fictitiously handled as control inputs, on the same level as the actual control $u(t)$. This structural result has been derived in a trivial way from geometric arguments (Theorems 5 and 6). The main contribution lies in the way this condition can be directly characterized from the transfer function matrices $G(s)$, $H(s)$ and $N(s)$ (Theorem 9). It should be interesting having a similar purely algebraic treatment for the case with internal stability, i.e., expressing $H''(s)$ in Theorem 6 without any geometric intermediary.

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