Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 3, 1998, pp. 1–?

An Adaptive Decoupling Compensator for Linear Systems Based on Periodic Multirate-Input Controllers*

K.G. Arvanitis

Abstract

An indirect adaptive controller synthesis algorithm is derived for the decoupling of linear multivariable time-invariant systems with unknown parameters using periodic multirate-input controllers. Such controllers contain a multirate sampling mechanism with different sampling period at each system input and a periodically varying modulating matrix function. The proposed adaptive algorithm is readily applicable to systems with different numbers of inputs and outputs, since the periodic multirate-input controllers used here reveal squaring-down capabilities. Furthermore, it does not rely on pole-zero cancellation, and therefore it can be readily applied to nonstably invertible plants and to diagonal reference models having arbitrary poles and zeros and relative degree. Moreover, the proposed adaptive algorithm estimates the unknown plant parameters (and hence the parameters of the desired modulating function) on line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period T_0. Finally, persistency of excitation of the continuous-time plant, and therefore parameter convergence, is provided without making any assumption concerning either the reference signals or the existence of specific convex sets in which the estimated parameters belong or, finally, the coprimeness of the polynomials describing the ARMA model. The only a priori knowledge needed to implement the algorithm, is the minimality of the continuous and sampled system, known order, and a set of structural indices, namely the locally minimum controllability indices of the continuous-time plant.

Key words: adaptive decoupling, sampled-data controllers, multirate sampling, periodically time-varying controllers

*Received February 1, 1996; received in final form February 18, 1997. Summary appeared in Volume 8, Number 3, 1998.

1 Introduction

Digital controllers containing periodically time-varying mechanisms are of particular interest, since they offer the possibility of solving, as an alternative to standard dynamic compensators, a wide class of important control problems. The reported results [1]-[14], may be grouped in two general categories, as follows: The first [1]-[8], [12], involves a sampling mechanism in which input or output samplers operate with a uniform multiplicity N (with $N = 1$, in some cases). The second [9]-[11], [13], [14], involves multirate sampling mechanisms, i.e., input or output samplers operate with a different sampling multiplicity at each plant input or output. Both categories have successfully been applied to the solution of several fundamental control problems, such as pole assignment [1], [3]-[5], [9]-[11], simultaneous and/or adaptive stabilization [6]-[8], exact model matching and decoupling [4], [7], [13], model reference adaptive control [12], [14], optimal control [2], [11], etc. From the so–far reported results, it is well recognized that periodically varying compensation provides several advantages over conventional time invariant feedback schemes (see [4], [14], for an extensive overview of these advantages).

This paper is devoted to the adaptive decoupling problem. Non-interacting control is an attractive control problem proposed in the literature and it has received much attention the last three decades, due to its theoretical and practical importance. The basic idea of the problem, originally proposed in [15], is to force the multi-input, multi-output (MIMO) system under control to behave like a set of single-input, single-output (SISO) systems, thus greatly facilitating the overall control strategy in cases where the designer is forced to control the different loops of a MIMO system individually or each output independently. A complete solution to the problem is given in [16]-[18]. Several important contributions based on different approaches were developed in the field and a very large number of papers have been reported on the subject (for an overview see [18]). The adaptive decoupling problem is also of great practical importance for obvious reasons. The first efforts to solve the problem appear in [19]-[21]. In [19], a priori knowledge of the non-minimum phase zeros of the controlled system is needed. Adaptive decoupling and prior knowledge of certain system characteristics are also discussed in [20]. In [21], the derived results do not guarantee exact decoupling and stability for general linear systems. Subsequently, different approaches have been applied to solve the problem [22]-[26]. In [22], [23], adaptive decoupling is achieved by the use of precompensators. In [22], it is necessary to factorize polynomial matrices into its Smith forms as well as to compute the adjoint of polynomial matrices. In [23], polynomial factorization of matrices is also necessary to isolate the unstable zeros of the system under control. In [24], an adaptive-decoupling controller synthesis algorithm is presented, which amounts to the design of a two-part controller, namely a pole-placement part to stabilize the system under control and a postcompensator part which is used to essentially decouple the system. Finally, in [25], [26], direct adaptive control schemes are presented to solve the adaptive decoupling problem for systems with stochastic disturbances. These adaptive schemes are based on the combination of the classical pole-zero placement strategy and the optimal strategy for self-tuning control. Since, in the present paper, our interest is focused in undisturbed plants, we next turn our attention to the approaches proposed in [22]-[24] and point out that, despite their differences, the basic features of these approaches are the following:

1. They reduce the problem to the solution of a matrix Diophantine equation, which is solved either on the basis of the widely used "certainty equivalence principle" (as for example in [23]) or on the basis of a recursive method (see [24], for details).

2. They contain a pole-zero placement part to avoid unstable pole-zero cancellations.

3. They apply only to systems with the same number of inputs and outputs.

4. Their convergence is established under the assumption that the input signals are persistently exciting. Nevertheless, recently, ways to remove this assumption are suggested. One such method is the decaying excitation procedure proposed in [27].

In the present paper the adaptive decoupling problem for linear time-invariant systems is attained on the basis of periodic multirate-input controllers (MRICs). MRICs were originally proposed by Araki and Hagiwara in [9], [10], in order to achieve arbitrary symmetric pole assignment in linear time-invariant systems. Such controllers contain a multirate sampling mechanism with different sampling period at each system input and a periodically varying modulating matrix function. MRICs can essentials be viewed as the special class of m-input, p-output multirate sampled-data control systems in which all output samplers operate with multiplicities 1 and the input samplers with multiplicities $\{N_1, \ldots, N_m\}$. Note that MRICs are the dual of multirate–output controllers (MROCs) presented in [28], in which input and output samplers have the reverse operation. It is worth noticing at this point that, although the inputs of the continuous-time plant are sampled in a multirate fashion, our aim here is to achieve adaptive decoupling control only at the sampling instants kT_0, associated with the fundamental period T_0, on the basis of which the output samplers operate.

To the authors' best knowledge, there are no results in the literature concerning the use of this type of multirate sampled-data controllers in order to achieve adaptive decoupling. The only partially relevant results are presented in [28], wherein adaptive decoupling control is attained on the basis of MROCs. The technique proposed here to solve the discrete adaptive decoupling control problem of continuous-time linear time-invariant systems, is motivated by some ideas regarding persistent excitation of a continuous-time plant, which are reported in [12], by the results of [14] and by some new results concerning the solution of the decoupling problem through exact model matching techniques. This technique results in a globally stable indirect adaptive control scheme, which estimates the unknown plant parameters (and hence the parameters of the desired modulating function) on line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by the fundamental sampling period T_0. It is pointed out that, the only a priori knowledge needed to implement the proposed algorithm, is the minimality of the continuous and sampled system, known order, and a set of structural indices, namely the locally minimum controllability indices (LMCI) of the continuous-time plant. Here, locally minimum controllability indices will be defined in the next section, but note that a typical example of LMCI is the set of "Kronecker invariants" of a controllable matrix pair.

The motivations for using periodic MRICs controlling linear system, are manifold. First of all, such controllers give more flexibility to the designer as compared to single-rate controllers, especially in cases where single-rate controllers cannot solve the problem considered. Moreover, in multirate-input controllers, sampling of the system outputs take place less often than sampling of the system inputs. Furthermore, multirate controllers maintain all benefits of periodic compensation over standard static or dynamic compensation. For an overview of these benefits, see [4], [10], [11], [13]. In particular, with regard to the adaptive decoupling problem treated in the present paper, we mention that the technique based on periodic MRICs has the following advantages over known techniques:

1. It reduces the solution of the problem to the solution of a simple non-homogeneous algebraic matrix equation, rather than a matrix Diophantine equation, as is needed in standard techniques.

2. It does not rely in pole-zero cancellation and hence it can be readily applied to solve the adaptive decoupling problem for nonstably invertible plants and for diagonal reference models having arbitrary poles and zeros and relative degree.

3. It is applicable in general to systems with different number of inputs And outputs, since the MRICs used here reveal squaring down capabilities.

4. It offers a solution to the problem of ensuring persistency of excitation of the continuous-time plant, without imposing any special requirement on the reference signal (except boundedness) and without making any assumption concerning either the existence of specific convex sets in which the estimated parameters belong or the coprimeness of the polynomials describing the ARMA model.

2 Preliminaries and Definition of the Problem

Consider the continuous-time, linear time-invariant multi-input, multi-output system having the following state-space representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\ y(t) = \mathbf{C}\mathbf{x}(t) \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbf{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbf{R}^m$ is the input vector and $\mathbf{y}(t) \in \mathbf{R}^p$ is the output vector and where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have appropriate dimensions.

With regard to the system (2.1) we make the following two assumptions:

Assumption 1 *(a) System (2.1) is controllable and observable and of known order n. (b) There are known integers n_i, $i \in \mathbf{J}_m$, $\mathbf{J}_m = \{1, 2, ..., m\}$, which comprise a set of locally minimum controllably indices of the pair $(\mathbf{A}, \mathbf{B})$.*

Assumption 2 *Let N_i, $i \in \mathbf{J}_m$ be positive integers. Also let $N = lcm\{N_1, ..., N_m\}$, where $lcm\{*, ..., *\}$ denotes the least common multiplier of the arguments quoted in the braces. Then, there is a sampling period $T_0 \in \mathbf{R}^+$, such that the discretized systems, obtained by sampling (2.1) with periods T_0 and $\tau = T_0/(6n-1)N$ and having the following matrix triplets*

$$(\mathbf{\Phi}, \hat{\mathbf{B}}, \mathbf{C}) \equiv (\exp(\mathbf{A}T_0), \int_0^{T_0} \exp(\mathbf{A}\lambda)\mathbf{B}d\lambda, \mathbf{C})$$

$$(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C}) \equiv (\exp(\mathbf{A}\tau), \int_0^{\tau} \exp(\mathbf{A}\lambda)\mathbf{B}d\lambda, \mathbf{C})$$

respectively are controllable and observable.

Except for this prior information, the matrix triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is arbitrary and unknown. It is mentioned that, no assumption is made here on the relative degree of the plant or its stable invertibility.

For a controllable matrix pair $(\mathbf{A}, \mathbf{B})$ with $\mathbf{B} = [\mathbf{b_1}\ \mathbf{b_2}\ \dots\ \mathbf{b_m}]$, its locally minimum controllability indices (LMCI) are a collection of m integers $\{n_1, n_2, \dots, n_m\}$, for which the following relationships simultaneously hold

$$\sum_{i=1}^{m} n_i = n \ \ and \ \ rank[\mathbf{b}_1 \ \ldots \ \mathbf{A}^{n_1-1}\mathbf{b}_1 \ \ldots \ \mathbf{b}_m \ \ldots \ \mathbf{A}^{n_m-1}\mathbf{b}_m] = n.$$

Note that, LMCI defined as above are also known as the "Kronecker invariants" or "Kronecker indexes" of the pair $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

The adaptive decoupling control problem treated in this paper is as follows: Given a discrete-time linear reference model $\mathbf{M}$ of the form

$$Z\{\mathbf{y}_m(kT_0)\} = \mathbf{H}_{\mathbf{M}}(z)\, Z\{\mathbf{w}(kT_0)\} \tag{2.2}$$

with

$$\mathbf{H}_{\mathbf{M}}(z) \stackrel{\triangle}{=} \mathbf{C}_{\mathbf{M}}(z\mathbf{I} - \mathbf{A}_{\mathbf{M}})^{-1}\mathbf{B}_{\mathbf{M}} \equiv \Delta(z) \stackrel{\triangle}{=} \underset{i=1,2,\ldots,p}{\text{diag}} \{\delta_i(z)\} \tag{2.3}$$

where $Z\{\bullet\}$ denotes the usual Z-transform, $\delta_i(z)$, for $i \in \mathbf{J}_p$, $\mathbf{J}_p = \{1, 2, ..., p\}$, are strictly causal rational functions for the desired diagonal reference model, whose denominators are strictly stable polynomials of the indeterminate z, and where $\mathbf{y}_{\mathbf{M}}(kT_0) \in \mathbf{R}^p$ is the output of the reference model and $\mathbf{w}(kT_0) \in \mathbf{R}^p$ is an arbitrary uniformly bounded reference sequence, then find a periodic multirate-input controller, which when applied to system (2.1), achieves discrete-time asymptotic model following, i.e.,

$$\lim_{k\to\infty} [\mathbf{y}(kT_0) - \mathbf{y}_M(kT_0)] = 0.$$

All signals in the control loop are bounded.

where $\mathbf{y}(kT_0)$ is the output of the plant evaluated at kT_0. It is pointed out that, no assumption is made here on the zeros and the relative degree of the model $\Delta(z)$, which may be arbitrary.

To solve the above problem, we next propose an indirect adaptive control scheme. In particular, we first solve the decoupling problem using a model matching technique, namely, we solve the problem of the exact matching of system (2.1) to the model (2.2), via MRICs. This is done is Section 3 and the corresponding control strategy is depicted in Figure 1. Next, using these results, the exact model matching problem is solved for the configuration depicted in Figure 2, wherein the periodic controller $\mathbf{F}(t)$ is with prespecified periodic behavior and where persistent excitations are introduced in the control loop for future identification purposes. This is accomplished in Section 4. It is remarked that the motivation for modifying the control strategy, as in Figure 2, is that it facilitates the derivation of the indirect adaptive control scheme sought. The derivation of the indirect adaptive scheme is presented in Section 5, where the global stability of the proposed scheme is also studied.

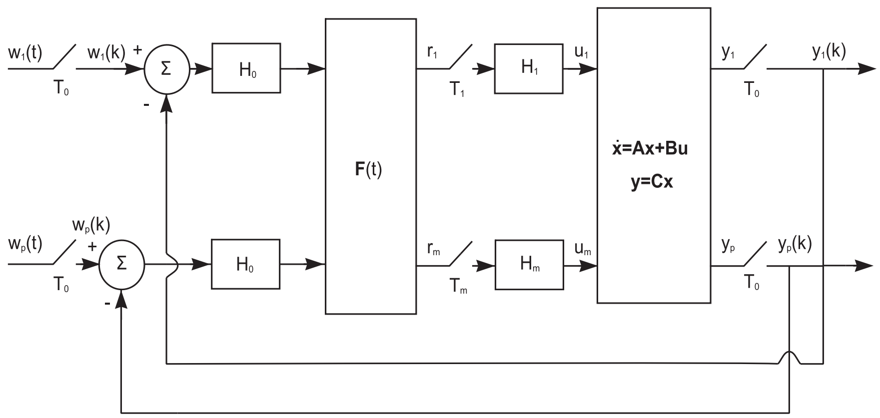

Figure 1: Control strategy in the nonadaptive case

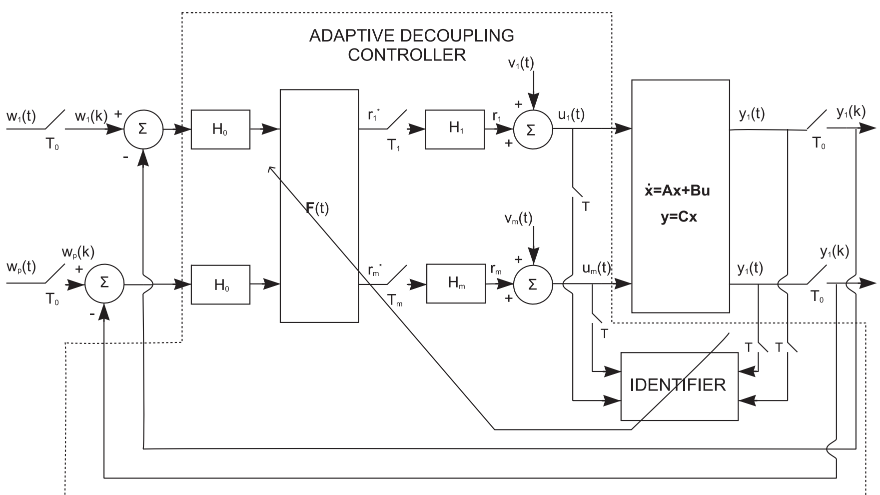

Figure 2: Structure of the adaptive control system

3 Solution of the Decoupling Problem for Known Systems

The control strategy proposed here to solve the decoupling problem in the case of known systems is depicted in Figure 1. With regard to the sampling mechanism, we assume that all samplers start simultaneously at $t = 0$. The sampling periods T_i have rational ratio, i.e., $T_i = T_0/N_i$, for $i \in \mathbf{J}_m$, where T_0 is the common sampling period and $N_i \in \mathbf{Z}^+$ are the *input multiplicities* of the sampling. The hold circuits H_i and H_0 are the zero order holds with holding times T_i and T_0, respectively. Let

$$N^* = \sum_{i=1}^{m} N_i \;\; , \;\; l_i = \frac{N}{N_i} \;\; , \;\; T_N = \frac{T_0}{N}.$$

The compensator $\mathbf{F}(t) \in \mathbf{R}^{m \times p}$ is a periodically time-varying controller with period T_0. That is

$$\mathbf{F}(t + T_0) = \mathbf{F}(t). \tag{3.1}$$

As it can be easily shown, the resulting closed-loop system is described by the following state space equations

$$\xi[(k+1)T_0] = (\mathbf{\Phi} - \mathbf{KC})\xi(kT_0) + \mathbf{Kw}(kT_0), \;\; \mathbf{y}(kT_0) = \mathbf{C}\xi(kT_0), \;\; for\, k \geq 0$$

where $\xi[kT_0] \in \mathbf{R}^n$ is the discrete measurement vector obtained by sampling $\mathbf{x}(t)$ with sampling period T_0 and where the matrix $\mathbf{K}$ is defined as

$$\mathbf{K} = \int_0^{T_0} \exp[\mathbf{A}(T_0 - \lambda)]\mathbf{BF}(\lambda)d\lambda. \tag{3.2}$$

System (2.1) can match system (2.3), using a periodic multirate-input controller of the form (3.1), with the common sampling period T_0, iff

$$\mathbf{H}_c(z) = \mathbf{C}(z\mathbf{I} - \mathbf{\Phi} + \mathbf{KC})^{-1}\mathbf{K} \;\equiv\; \mathbf{H}_{\mathbf{M}}(z) = \Delta(z) = \operatorname*{diag}_{i=1,2,\ldots,p} \{\delta_i(z)\} \tag{3.3}$$

In what follows, a new technique is presented for solving (3.3) with regard to the matrix $\mathbf{K}$. The proposed technique is as follows: Observe first that, from relation (3.3), we can easily obtain

$$\mathbf{C}(z\mathbf{I} - \mathbf{\Phi} + \mathbf{KC})^{-1}\mathbf{k_i} = \delta_i(z)\epsilon_i \;\; , \;\; for \;\; i \in \mathbf{J}_p \tag{3.4}$$

where, $\mathbf{k_i}$ is the ith column of the matrix $\mathbf{K}$ and where $\epsilon_i \in \mathbf{R}^p$, for $i \in \mathbf{J}_p$, is the column vector whose elements are all zeros except for a unity appearing in the ith position.

Next, expand both sides of (3.4) in a series of negative powers of z, to yield

$$\sum_{j=1}^{\infty} z^{-j}\mathbf{C}(\mathbf{\Phi}-\mathbf{KC})^{j-1}\mathbf{k_i} = \sum_{i=1}^{\infty} z^{-j}(\delta_i)_j\epsilon_i \quad , \tag{3.5}$$

$$for \ \ i \in \mathbf{J}_p \ \ and \ \ for \ \ j \geq 1$$

where $(\delta_i)_j$, for $i \in \mathbf{J}_p$ and for $j \geq 1$, are the coefficients of the expansion of $\delta_i(z)$ in a series of negative powers of z. Equating coefficients of like powers of z^{-1} in (3.5), we obtain

$$\mathbf{C}(\mathbf{\Phi}-\mathbf{KC})^{j-1}\mathbf{k_i} = (\delta_i)_j\epsilon_i \quad , \quad for \ \ j \geq 1. \tag{3.6}$$

In (3.6), it is sufficient to keep only the first $2n$ equations, since for the realization of a strictly causal rational function of the intermediate z, only the first $2n$ coefficients of its respective expansion in a Laurent series are needed (see [29]), for a detailed analysis of this issue). Then, relation (3.6) reduces to the following set of equations

$$\mathbf{C}(\mathbf{\Phi}-\mathbf{KC})^{j-1}\mathbf{k_i} = (\delta_i)_j\epsilon_i \quad , \quad for \ \ j = 1, 2, ..., 2n. \tag{3.7}$$

We next manipulate (3.7), in order to obtain a more useful result. To this end, let $(\gamma_i)_1 = (\delta_i)_1$. Then relation (3.7), for $j = 1$, yields

$$\mathbf{Ck_i} = (\gamma_i)_1\epsilon_i. \tag{3.8}$$

Observe now that on the basis of (3.8), the second equation of the system (3.7) takes the form

$$\mathbf{C}(\mathbf{\Phi}-\mathbf{KC})\mathbf{k_i} = \mathbf{C\Phi k_i} - \mathbf{CKCk_i} = \mathbf{C\Phi k_i} - \operatorname*{diag}_{i=1,2,\ldots,p}\{(\delta_i)_1\}(\gamma_i)_1\epsilon_i =$$

$$\mathbf{C\Phi k_i} - (\delta_i)_1(\gamma_i)_1\epsilon_i = (\delta_i)_2\epsilon_i.$$

Defining $(\gamma_i)_2 = (\delta_i)_2 + (\delta_i)_1(\gamma_i)_1$, the foregoing relation can also be written as

$$\mathbf{C\Phi k_i} = (\gamma_i)_2\epsilon_i. \tag{3.9}$$

The third equation of system (3.7), can further be written as

$$\mathbf{C\Phi}^2\mathbf{k}_i - \mathbf{CKC\Phi k_i} - \mathbf{C\Phi KCk_i} + \mathbf{CKCKCk_i} = (\delta_i)_3\epsilon_i. \tag{3.10}$$

On the basis of (3.8) and (3.9), the left-hand-side of (3.10) can also be written as

$$\mathbf{C\Phi}^2\mathbf{k}_i - (\delta_i)_1(\gamma_i)_2\epsilon_i - (\gamma_i)_2(\delta_i)_1\epsilon_i + [(\delta_i)_1]^3\epsilon_i$$

$$= \mathbf{C\Phi}^2\mathbf{k}_i - (\delta_i)_1(\gamma_i)_2\epsilon_i - [(\delta_i)_2 + (\gamma_i)_1(\delta_i)_1](\delta_i)_1\epsilon_i + [(\delta_i)_1]^3\epsilon_i$$

$$= \mathbf{C\Phi}^2\mathbf{k}_i - (\delta_i)_1(\gamma_i)_2\epsilon_i - [(\delta_i)_2 + [(\delta_i)_1]^2](\delta_i)_1\epsilon_i + [(\delta_i)_1]^3\epsilon_i$$

$$= \mathbf{C\Phi}^2\mathbf{k}_i - (\delta_i)_1(\gamma_i)_2\epsilon_i - (\delta_i)_2(\delta_i)_1\epsilon_i$$

$$= \mathbf{C\Phi}^2\mathbf{k}_i - (\delta_i)_1(\gamma_i)_2\epsilon_i - (\delta_i)_2(\gamma_i)_1\epsilon_i. \tag{3.11}$$

Introducing (3.11) in (3.10) and defining

$$(\gamma_i)_3 = (\delta_i)_3 + (\delta_i)_2(\gamma_i)_1 + (\delta_i)_1(\gamma_i)_2,$$

we readily obtain

$$\mathbf{C\Phi^{j-1}k_i} = (\gamma_i)_3\epsilon_i$$

By repeatedly using the above algorithm, one can easily replace the jth equation of the system (3.7) (for $j = 1, 2, \ldots, 2n$), by its equivalent equation having the form

$$\mathbf{C\Phi^{j-1}k_i} = (\gamma_i)_j\epsilon_i \ , \ for \ \ j = 1, 2, \ldots, 2n \tag{3.12}$$

where

$$(\gamma_i)_1 = (\delta_i)_1 \ \ and \ \ (\gamma_i)_{j+1} = (\delta_i)_{j+1} + \sum_{r=1}^{j}(\delta_i)_r(\gamma_i)_{j-r+1}.$$

Relation (3.12), can be written in a compact vector-matrix form as follows

$$\mathbf{Mk_i} = \mathbf{g}_i \ \ , \ \ for \ \ i \in \mathbf{J}_p \tag{3.13}$$

where the matrix $\mathbf{M} \in \mathbf{R}^{2np\times p}$ and the column vector $\mathbf{g}_i \in \mathbf{R}^{2np}$ have the following forms

$$\mathbf{M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C\Phi} \\ \vdots \\ \mathbf{C\Phi}^{2n-1} \end{bmatrix} \ , \ \ \mathbf{g}_i = \begin{bmatrix} (\gamma_i)_1\epsilon_i \\ (\gamma_i)_2\epsilon_i \\ \vdots \\ (\gamma_i)_{2n}\epsilon_i \end{bmatrix} \ , \ \ for \ \ i \in \mathbf{J}_p. \tag{3.14}$$

Relation (3.13) is a linear non-homogeneous algebraic system of equations. The matrix $\mathbf{M}$ and the column vector $\mathbf{g}_i$ are known and depend upon the matrices $\mathbf{C}$ and $\mathbf{\Phi}$ and upon the Markov parameters $(\delta_i)_j$, for $i \in \mathbf{J}_p$ and for $j = 1, 2, \ldots, 2n$, of the desired diagonal model, respectively. Clearly, the solution of the decoupling problem (3.3) is now reduced to that of solving (3.13), with respect to $\mathbf{k_i}$, $i \in \mathbf{J}_p$. With regard to the solution of (3.13), we point out that, since by Assumption 2.2 the pair $(\mathbf{\Phi}, \mathbf{C})$ is observable, the matrix $\mathbf{M}$ has full column rank equal to n. Consequently, relation (3.13) has a solution if and only if

$$rank[\mathbf{M} \;\; \mathbf{g}_i] = n \tag{3.15}$$

or equivalently if $\mathbf{g}_i$ is a linear combination of the columns of matrix $\mathbf{M}$. Furthermore, a solution of (3.13) is the following

$$\mathbf{k_i} \equiv \mathbf{q}_i = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{g}_i. \tag{3.16}$$

Hence, the solution for the matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{Q}_s = [\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \ldots \;\; \mathbf{q}_p]. \tag{3.17}$$

Using the matrix $\mathbf{K}$ as specified by (3.17), we can readily determine the controller matrix $\mathbf{F}$(t), by solving (3.2). Under Assumption 1, on the controllability of the pair $(\mathbf{A}, \mathbf{B})$, a solution of (3.2) is the following [7]

$$\mathbf{F}(t) = \mathbf{B}^T \exp[\mathbf{A}^T(T_0 - t)]\mathbf{W}^{-1}(\mathbf{A}, \mathbf{B}, T_0)\mathbf{K} \tag{3.18}$$

where $\mathbf{W}(\mathbf{A}, \mathbf{B}, T_0)$ is the controllability Grammian on $[0, T_0]$ of the pair $(\mathbf{A}, \mathbf{B})$, which has the form

$$\mathbf{W}(\mathbf{A}, \mathbf{B}, T_0) = \int_0^{T_0} \exp[\mathbf{A}(T_0 - \lambda)]\mathbf{B}\mathbf{B}^T \exp[\mathbf{A}^T(T_0 - \lambda)]d\lambda.$$

Note that, the controllability Grammian $\mathbf{W}(\mathbf{A}, \mathbf{B}, T_0)$ is nonsingular and hence a solution of (3.2) of the form (3.18) exists if the pair $(\mathbf{A}, \mathbf{B})$ is controllable.

4 A Solution of the Decoupling Problem Appropriate for the Adaptive Case

In order to obtain a solution of the decoupling problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify in the sequel the control strategy of Figure 1 as it is depicted in Figure 2. In particular, we focus our attention on the special class of the time-varying T_0-periodic controllers $\mathbf{F}(t)$, for which every element of $\mathbf{F}(t)$, denoted by $f_{ij}(t)$, is piecewise constant over intervals of length T_i, i.e.,

$$f_{ij}(t) = f_{ij,\mu} \quad \forall t \in [\mu T_i, (\mu + 1)T_i) \;\; , \;\; \mu = 0, 1, \ldots, N_i - 1. \tag{4.1}$$

The persistent excitation signals $\nu_i(t)$, $\forall i \in \mathbf{J}_m$ are defined as

$$\nu_i(t) = \mathbf{d}_i^T(t)\mathbf{v}_i \ \ , \ \ \mathbf{d}_i^T(t) = [(d_i)_0(t), \ldots, (d_i)_{N_i-1}(t)]. \tag{4.2}$$

Here, $\mathbf{d}_i(t)$ is the T_i-periodic vector function with elements having the form

$$(d_i)_q(t) = (d_i)_{q,\mu} \ \ , \ \ for \ \ t \in [\mu T_i, (\mu+1)T_i) \tag{4.3}$$
$$q = 0, 1, \ldots, N_i - 1 \ \ , \ \ \mu = 0, 1, \ldots, N_i - 1$$

where $(d_i)_{q,\mu}$ are constant taking the following values

$$(d_i)_{q,\mu} = \begin{cases} 1, & for \ \ \mu = q \\ 0, & for \ \ \mu \neq q \end{cases} \tag{4.4}$$

and where $\mathbf{v}_i$ is as yet unknown. It is worth noticing that the additive term $\nu_i(t) = \mathbf{d}_i^T(t)\mathbf{v}_i$, $\forall i \in \mathbf{J}_m$, in each one of the inputs of the continuous-time system, are used only for identification purposes and as it will be shown later, they are selected so that they will not influence the decoupling problem.

We are now able to establish the following Lemma.

Lemma 1 *Consider the controllable and observable system of the form (2.1), controlled by a periodic multirate-input controller of the form (4.1). Furthermore, consider that persistent excitation signals of the form (4.2)-(4.4) are introduced to each input of the system. Then, the sampled closed-loop system takes the form*

$$\xi[(k+1)T_0] = (\mathbf{\Phi} - \hat{\mathbf{B}}\hat{\mathbf{F}}\mathbf{C})\xi(kT_0) + \hat{\mathbf{B}}\hat{\mathbf{F}}\mathbf{w}(kT_0) + \mathbf{B}^*\mathbf{v} \ , \tag{4.5}$$

$$\mathbf{y}(kT_0) = \mathbf{C}\xi(kT_0), \ \ for \ \ k \geq 0$$

where

$$\hat{\mathbf{B}} \triangleq [\hat{\mathbf{b}}_1 \ \ldots \ \hat{\mathbf{A}}_1^{N_1-1}\hat{\mathbf{b}}_1 \ \ldots \ \hat{\mathbf{b}}_m \ \ldots \ \hat{\mathbf{A}}_m^{N_m-1}\hat{\mathbf{b}}_m] \tag{4.6}$$

$$\hat{\mathbf{A}} \triangleq \exp(\mathbf{A}T_i) \equiv \exp(\mathbf{A}l_iT_N) \ \ ,$$

$$\hat{\mathbf{b}}_i \triangleq \int_0^{T_i} \exp(\mathbf{A}\lambda)\mathbf{b}_i d\lambda \equiv \int_0^{l_iT_N} \exp(\mathbf{A}\lambda)\mathbf{b}_i d\lambda \tag{4.7}$$

$$\mathbf{B}^* = \hat{\mathbf{B}}\mathbf{T} \ \ , \ \ \mathbf{T} = \begin{bmatrix} \mathbf{T_1} \\ \mathbf{T_2} \\ \vdots \\ \mathbf{T_m} \end{bmatrix} \ , \ \ \mathbf{T}_j = \begin{bmatrix} \mathbf{e}_{\sigma_j} \\ \mathbf{e}_{\sigma_j-1} \\ \vdots \\ \mathbf{e}_{\sigma_j-N_j+1} \end{bmatrix} \tag{4.8}$$

and where the $m \times p$ block matrix $\hat{\mathbf{F}}$ and the column vector $\mathbf{v} \in \mathbf{R}^{N^*}$ have the forms

$$\hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{f}}_{\mathbf{11}} & \cdots & \hat{\mathbf{f}}_{\mathbf{1p}} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{f}}_{\mathbf{m1}} & \cdots & \hat{\mathbf{f}}_{\mathbf{mp}} \end{bmatrix} , \quad \hat{\mathbf{f}}_{ij} = \begin{bmatrix} f_{ij,N_i-1} \\ \vdots \\ f_{ij,0} \end{bmatrix} \tag{4.9}$$

$$\mathbf{v} = [\mathbf{v}_1^T \ \mathbf{v}_2^T \ \ldots \ \mathbf{v}_m^T]^T \tag{4.10}$$

while $\sigma_j = \sum_{\kappa=1}^{j} N_\kappa$, where in general, the vector $\mathbf{e}_i \in \mathbf{R}^{N^*}$ is the row vector whose elements are zeros except for a unity appearing in the ith position.

Proof: To show that the sampled closed-loop system takes the form (4.5), we start by discretizing system (2.1) with sampling period T_0. This operation yields

$$\xi[(k+1)T_0] = \mathbf{\Phi}\xi(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp\{\mathbf{A}[(k+1)T_0 - \lambda]\}\mathbf{B}\mathbf{u}(\lambda)d\lambda. \tag{4.11}$$

Observing that $u_i(t) = r_i(t) + \mathbf{d}_i^T(t)\mathbf{v}_i$ and taking into account the structure of the control system in Figure 2, we obtain

$$u_i(t) = \mathbf{f}_i^T(t)\mathbf{e}(kT_0) + \mathbf{d}_t^T(t)\mathbf{v}_i \ , \quad for\ t \in [\mu T_i\, ,\ (\mu+1)T_i) \tag{4.12}$$

where $\mathbf{f}_i^T(t)$ is the ith row of the controller matrix $\mathbf{F}(t)$ and $\mathbf{e}(kT_0)$ is given by

$$\mathbf{e}(kT_0) = \mathbf{w}(kT_0) - \mathbf{y}(kT_0) = \mathbf{w}(kT_0) - \mathbf{C}\xi(kT_0). \tag{4.13}$$

Combining relations (4.11)-(4.13), we obtain the following relationship

$$\xi[(k+1)T_0] = (\mathbf{\Phi} - \mathbf{KC})\xi(kT_0) + \mathbf{Kw}(kT_0) + \mathbf{\Gamma v} \tag{4.14}$$

where

$$\mathbf{\Gamma} = \int_{kT_0}^{(k+1)T_0} \exp\{\mathbf{A}[(k+1)T_0 - \lambda]\}\mathbf{BD}(\lambda)d\lambda \ , \quad \mathbf{D}(t) \triangleq \underset{i\in\mathbf{J}_m}{\text{blockdiag}}\{\mathbf{d}_i^T(t)\}$$

Now, partition $\mathbf{\Gamma}$ as follows

$$\mathbf{\Gamma} = [\mathbf{\Gamma}_1 \ \mathbf{\Gamma}_2 \ \ldots \ \mathbf{\Gamma}_m].$$

Then, the $(q+1)$th column of the matrix $\mathbf{\Gamma}_i$ for $i \in \mathbf{J}_m$, denoted by $(\mathbf{\Gamma}_i)_{q+1}$, for $q = 0, 1, \ldots, N_i - 1$, can be expressed as

$$(\mathbf{\Gamma}_i)_{q+1} = \int_0^{T_0} \exp[\mathbf{A}(T_0 - \lambda)]\mathbf{b}_i(d_i)_q(\lambda)d\lambda \ , \tag{4.15}$$

$$for\ q = 0, 1, \ldots, N_i - 1$$

Introducing relations (4.2) and (4.4) in (4.15), we obtain

$$(\mathbf{\Gamma}_i)_{q+1} = \sum_{\mu=0}^{N_i-1} \int_{\mu T_i}^{(\mu+1)T_i} \exp[\mathbf{A}(T_0 - \lambda)\mathbf{b}_i(d_i)_{q,\mu} d\lambda] \quad , \tag{4.16}$$

$$for\ q = 0, 1, \ldots, N_i - 1$$

Relation (4.16) may further be written as

$$(\mathbf{\Gamma}_i)_{q+1} = \sum_{\mu=0}^{N_i-1} (d_i)_{q,\mu} \exp[\mathbf{A}(N_i - \mu - 1)T_i] \int_0^{T_i} \exp[\mathbf{A}(T_i - \lambda)]\mathbf{b}_i d\lambda$$

$$= (\sum_{\varrho=1}^{N_i} (d_i)_{q,N_i-\varrho} \hat{\mathbf{A}}_i^{\varrho-1})\hat{\mathbf{b}}_i.$$

Making use of relation (4.4), we arrive at the following relationship

$$(\mathbf{\Gamma}_i)_{q+1} = \hat{\mathbf{A}}_i^{N_i-q-1}\hat{\mathbf{b}}_i.$$

Clearly $\mathbf{\Gamma} \equiv \mathbf{B}^*$. Application of the above algorithm to the first two terms of (4.14) yields $\mathbf{K} \equiv \hat{\mathbf{B}}\hat{\mathbf{F}}$. This completes the proof of the lemma. □

Thus far, we have established that the decoupling controller matrix $\mathbf{K}$ is related to the matrix $\hat{\mathbf{F}}$ via the relation $\mathbf{K} \equiv \hat{\mathbf{B}}\hat{\mathbf{F}}$. It remains to determine $\hat{\mathbf{F}}$. To this end, we need the following result, whose proof is given in [10].

Lemma 2 *Let $(\mathbf{A}, \mathbf{B})$ be a controllable pair. Let also $n_i, i \in \mathbf{J}_m$ be a set of locally minimum controllability indices of the pair $(\mathbf{A}, \mathbf{B})$. Define an analytic function $\psi(T_N)$ by*

$$\psi(T_N) = det[\hat{\mathbf{b}}_1\ \ldots\ \hat{\mathbf{A}}_1^{n_1-1}\hat{\mathbf{b}}_1\ \ldots\ \hat{\mathbf{b}}_m\ \ldots\ \hat{\mathbf{A}}_m^{n_m-1}\hat{\mathbf{b}}_m].$$

Then the set of zeros of $\psi(T_N)$ does not have any limiting points except infinity, and therefore, $\psi(T_N)$ is not equal to zero for almost all T_N (i.e.,, in a finite interval $[T_N^1, T_N^2]$, there are at most a finite number of points such that $\psi(T_N) = 0$).

Applying Lemma 2, we can conclude that the matrix $\hat{\mathbf{S}}$ of the form

$$\hat{\mathbf{S}} = [\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_1^{n_1-1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{b}}_m \ldots \hat{\mathbf{A}}_m^{n_m-1}\hat{\mathbf{b}}_m] \tag{4.17}$$

is nonsingular for almost all $T_N \in [T_N^1, T_N^2]$. Furthermore, if the input multiplicities of the sampling N_i are chosen such that $N_i \geq n_i$, $i \in \mathbf{J}_m$ then, the matrices $\hat{\mathbf{B}}$ and $\mathbf{B}^*$ have full row rank n for almost all $T_N \in [T_N^1, T_N^2]$.

Now, let $\mathbf{E} \in \mathbf{R}^{N^* \times N^*}$ be the nonsingular permutation matrix with the property $\mathbf{E}^{-1} \equiv \mathbf{E}^T$, having the form

$$\mathbf{E} = [\mathbf{E}_1 \;\; \mathbf{E}_2]^T$$

where

$$\mathbf{E}_1 = [\epsilon_1 \; \epsilon_2 \ldots \epsilon_{n_1} \; \epsilon_{N_1+1} \; \epsilon_{N_1+2} \ldots \epsilon_{N_1+n_2} \ldots \epsilon_{N^*-N_m+1} \; \epsilon_{N^*-N_m+2} \ldots \epsilon_{N^*-N_m+n_m}]$$

and

$$\mathbf{E}_2 = [\epsilon_{n_1+1} \ldots \epsilon_{N_1} \; \epsilon_{N_1+n_2+1} \ldots \epsilon_{N_1+N_2} \ldots \epsilon_{N^*-N_m+n_m+1} \ldots \epsilon_{N^*}]$$

where, in general, $\epsilon_j \in \mathbf{R}^{N^*}$ is the column vector whose elements are zeros except for a unity appearing in the jth position. Also, let

$$\tilde{\mathbf{B}} \triangleq \hat{\mathbf{B}}\mathbf{E}^{-1} = [\hat{\mathbf{S}} \; \hat{\mathbf{Q}}]$$

where the matrix $\hat{\mathbf{S}}$ is defined by (4.17) and matrix $\hat{\mathbf{Q}}$ is given by

$$\hat{\mathbf{Q}} = [\hat{\mathbf{A}}_1^{n_1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_1^{N_1-1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_m^{n_m}\hat{\mathbf{b}}_m \ldots \hat{\mathbf{A}}_m^{N_m-1}\hat{\mathbf{b}}_m].$$

Furthermore, let $\mathbf{\Delta} \in \mathbf{R}^{N^* \times N^*}$ be the nonsingular permutation matrix with the property $\mathbf{\Delta}^{-1} \equiv \mathbf{\Delta}^T$, having the form

$$\mathbf{\Delta} = [\mathbf{\Delta}_1 \;\; \mathbf{\Delta}_2 \;\; \mathbf{\Delta}_3]^T$$

where

$$\mathbf{\Delta}_1 = [\epsilon_{N_1-n_1+1} \ldots \epsilon_{N_1} \; \epsilon_{N_1+N_2-n_2+1} \ldots \epsilon_{N_1+N_2} \ldots \epsilon_{N^*-N_m+1} \ldots \epsilon_{N^*}]$$

$$\mathbf{\Delta}_2 = [\epsilon_{N_1-n_1} \; \epsilon_{N_1+N_2-n_2} \ldots \epsilon_{N^*-N_m}]$$

$$\mathbf{\Delta}_3 = [\epsilon_1 \ldots \epsilon_{N_1-n_1-1} \; \epsilon_{N_1+1} \ldots \epsilon_{N_1+N_2-n_2-1} \ldots \epsilon_{N^*-N_m+1} \ldots \epsilon_{N^*-n_m-1}].$$

Finally, let

$$\tilde{\mathbf{B}}^* \triangleq \mathbf{B}^*\mathbf{\Delta}^{-1} \equiv [\hat{\mathbf{S}}^* \quad \hat{\mathbf{A}}_1^{n_1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_m^{n_m}\hat{\mathbf{b}}_m \quad \hat{\mathbf{Q}}^*]$$

where

$$\hat{\mathbf{S}}^* = [\hat{\mathbf{A}}_1^{n_1-1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_m^{n_m-1}\hat{\mathbf{b}}_m \ldots \hat{\mathbf{b}}_m] \tag{4.18}$$

$$\hat{\mathbf{Q}}^* = [\hat{\mathbf{A}}_1^{N_1-1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_1^{n_1+1}\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{A}}_m^{N_m-1}\hat{\mathbf{b}}_m \ldots \hat{\mathbf{A}}_m^{n_m+1}\hat{\mathbf{b}}_m].$$

Using these definitions, it is plausible to determine $\hat{\mathbf{F}}$ by mere inspection, as

$$\hat{\mathbf{F}} = \mathbf{E}^T \begin{bmatrix} \hat{\mathbf{S}}^{-1}\mathbf{Q}_s \\ 0 \end{bmatrix}. \tag{4.19}$$

It only remains to determine the appropriate vector $\mathbf{v}$ which guarantees that the decoupling problem will not be dependent on the vector $\mathbf{v}$. In other words

$$\mathbf{v} \in ker\mathbf{B}^* \;\; or \;\; \mathbf{B}^*\mathbf{v} = 0.$$

An obvious selection of such $\mathbf{v}$ obtained also by inspection is the following

$$\mathbf{v} = \mathbf{\Delta}^T \begin{bmatrix} -\hat{\mathbf{S}}^{*-1}(\hat{\mathbf{A}}_1^{n_1}\hat{\mathbf{b}}_1 + \ldots + \hat{\mathbf{A}}_m^{n_m}\hat{\mathbf{b}}_m) \\ \zeta \\ \mathbf{0}_{N^*-n-m} \end{bmatrix} \tag{4.20}$$

where $\zeta \in \mathbf{R}^m$ is the column vector whose elements are all equal to 1.

It is noted that the N^*-dimensional column vector $\mathbf{v}$, even though does not affect the discrete decoupling problem, it provides persistent excitation useful for the consistent identification of the system, as will be shown in the following Section.

Clearly, the multirate controller matrix $\mathbf{F}(t)$ of Figure 2 can readily be determined by making use of relations (4.1), (4.9), (4.19) and (3.17). More precisely, the ith row $\mathbf{f}_i^T(t)$ of the matrix $\mathbf{F}(t)$ and the ith block row of the matrix $\hat{\mathbf{F}}$ are interrelated as

$$\mathbf{f}_i^T(t) = [f_{i1}(t) \ldots f_{ip}(t)] = e_{N_1-\mu}[\hat{\mathbf{f}}_{i1} \ldots \hat{\mathbf{f}}_{ip}] \;,\;\; \forall \frac{\mu T_0}{N_i} \leq t < \frac{(\mu+1)T_0}{N_i} \tag{4.21}$$

for $i \in \mathbf{J}_m$ and for $\mu = 0, 1, \ldots, N_i - 1$, where $e_{N_i-\mu} \in \mathbf{R}^{N_i}$ is the row vector defined as $e_{N_i-\mu} = \epsilon_{N_i-\mu}^T$. Note that, the controller matrix $\mathbf{F}(t)$, as specified by (4.21), is largely affected by the multirate mechanism, while the controller matrix $\mathbf{F}(t)$ as specified by relation (3.18) is not. Furthermore, the introduction of the excitation signals $v_i(t)$ in the control loop, greatly facilitates the consistent estimation of the plant parameters in the case of unknown systems. For these reasons, the control strategy of Figure 2 is more appropriate than the control strategy of Figure 1 for the development of the indirect adaptive control scheme presented on the following Section.

5 Control Strategy for the Adaptive Case

The control scheme presented in Section 4 has a corresponding scheme in the case where the system is unknown. For this case, the control strategy is largely based on the computation of the matrix $\hat{\mathbf{F}}$ and of the vector $\mathbf{v}$ from estimates of the plant parameters, and results in a globally stable closed-loop system whose output asymptotically follows the output of the desired decoupled model.

5.1 Plant parameters estimation algorithm

The algorithm proposed here for estimating the unknown plant parameters is as follows: System (2.1), discretized with sampling period $\tau = T_0/(6n-1)N$, takes the form

$$\xi[(\nu+1)\tau] = \mathbf{\Phi}_\tau \xi(\nu\tau) + \mathbf{B}_\tau \mathbf{u}(\nu\tau) \ \ , \ \ \mathbf{y}(\nu\tau) = \mathbf{C}\xi(\nu\tau) \ \ , \ \ \nu \geq 0 \tag{5.1}$$

where

$$\mathbf{\Phi}_\tau = \exp(\mathbf{A}\tau) \ \ , \ \ \mathbf{B}_\tau = \int_0^\tau \exp(\mathbf{A}\lambda)\mathbf{B}d\lambda.$$

Clearly, $\mathbf{u}(\nu\tau)$ takes constant values for $\nu\tau \in [\varrho T_N, \ (\varrho+1)T_N], \ \varrho \geq 0$. This can be easily shown by taking into account the action of the proposed controller. Hence, iterating relation (5.1) $6n-1$ times, we obtain

$$\xi[(m+1)T_N] = \mathbf{\Phi}_{T_N}\xi(mT_N) + \mathbf{B}_{T_N}\mathbf{u}(mT_N) \ \ , \ \ m \geq 0$$

where

$$\mathbf{\Phi}_{T_N} = (\Phi_\tau)^{6n-1} \ \ , \ \ \mathbf{B}_{T_N} = \sum_{\varrho=0}^{6n-2} \mathbf{\Phi}_\tau^\varrho \mathbf{B}_\tau \tag{5.2}$$

Using the same argument, we can easily conclude that

$$\hat{\mathbf{A}}_i = \Phi_{T_N}^{l_i} \ \ , \ \ \hat{\mathbf{b}}_i = \sum_{\varrho=0}^{l_i-1} \mathbf{\Phi}_{T_N}^\varrho (\mathbf{B}_{T_N})_i \tag{5.3}$$

where $(\mathbf{B}_{T_N})_i$ is the ith column of the matrix $\mathbf{B}_{T_N}$. Introducing relation (5.2) in (5.3), yields

$$\hat{\mathbf{A}}_i = \Phi_\tau^{(6n-1)l_i} \ \ , \ \ \hat{\mathbf{b}}_i = \sum_{j=0}^{l_i-1} (\mathbf{\Phi}_\tau)^{(6n-1)j} (\sum_{\varrho=0}^{6n-2} \mathbf{\Phi}_\tau^\varrho \mathbf{B}_\tau)_i. \tag{5.4}$$

Moreover, the matrix $\mathbf{\Phi}$ can be written as

$$\mathbf{\Phi} = \hat{\mathbf{A}}_i^{N_i} = \mathbf{\Phi}_{T_N}^N = (\Phi_\tau)^{(6n-1)N}. \tag{5.5}$$

Therefore, $\mathbf{\Phi}$, $\hat{\mathbf{A}}_i$ and $\hat{\mathbf{b}}_i$ (which are the only matrices involved in computing $\hat{\mathbf{F}}$ and $\mathbf{v}$) can be computed on the basis of $\mathbf{\Phi}_\tau$ and $\mathbf{B}_\tau$. For this reason, in what follows our aim will be the estimation of the matrix triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$. To this end, let the matrix $\mathbf{\Omega}$ be defined as

$$\mathbf{\Omega} = \{\mathbf{\Omega}_{ij}\}_{j=1,2,\ldots,n}^{i=1,2,\ldots,n} \ \ , \ \ \mathbf{\Omega}_{ij} = \mathbf{C}\mathbf{\Phi}_\tau^{i+j-2}\mathbf{B}_\tau. \tag{5.6}$$

Clearly, if one establishes estimates of the matrix $\mathbf{\Omega}$, then one may easily compute the desired matrix triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$, using anyone of the minimal realisation algorithms reported in the literature (see for example those reported in [29]-[31]). To estimate matrix $\mathbf{\Omega}$, one must resort to an input-output representation (also called ARMA representation) of system (5.1). This representation is summarized in the following Theorem:

Theorem 1 *Suppose that there is a sampling period $T_0 \in \mathbf{R}^+$ and input multiplicities of the sampling N_i, $i \in \mathbf{J}_m$, such that the system (5.1), obtained by sampling the controllable and observable system (2.1), is also controllable and observable. Then, an alternative representation of system (5.1) is given by*

$$\mathbf{\Psi}(\nu\tau) = \mathbf{J}_1\mathbf{\Psi}[(\nu-2n)\tau] + \mathbf{J}_2\mathbf{W}(\nu\tau) + \mathbf{V}\mathbf{W}[(\nu-n)\tau] + \mathbf{V}^*\mathbf{W}[(\nu-2n)\tau] \tag{5.7}$$

where

$$\mathbf{\Psi}(\nu\tau) = \begin{bmatrix} \mathbf{y}[(\nu-n+1)\tau] \\ \mathbf{y}[(\nu-n+2)\tau] \\ \vdots \\ \mathbf{y}(\nu\tau) \end{bmatrix}, \quad \mathbf{\Psi}[(\nu-2n)\tau] = \begin{bmatrix} \mathbf{y}[(\nu-3n+1)\tau] \\ \mathbf{y}[(\nu-3n+2)\tau] \\ \vdots \\ \mathbf{y}[(\nu-2n)\tau] \end{bmatrix}$$

$$\mathbf{W}(\nu\tau) = \begin{bmatrix} \mathbf{u}[(\nu-n+1)\tau] \\ \mathbf{u}[(\nu-n+2)\tau] \\ \vdots \\ \mathbf{u}(\nu\tau) \end{bmatrix} \tag{5.8}$$

$$\mathbf{W}[(\nu-\mathrm{n})\tau] = \begin{bmatrix} \mathbf{u}[(\nu-2n+1)\tau] \\ \mathbf{u}[(\nu-2n+2)\tau] \\ \vdots \\ \mathbf{u}[(\nu-n)\tau] \end{bmatrix}, \mathbf{W}[(\nu-2\mathrm{n})\tau] = \begin{bmatrix} \mathbf{u}[(\nu-3n+1)\tau] \\ \mathbf{u}[(\nu-3n+2)\tau] \\ \vdots \\ \mathbf{u}[(\nu-2n)\tau] \end{bmatrix} \tag{5.9}$$

$$\mathbf{J}_1 = \mathbf{\Xi}^{*-1}\begin{bmatrix} \hat{\mathbf{J}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{\Xi}^*, \ \mathbf{J}_2 = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\mathbf{B}_\tau & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{C}\mathbf{\Phi}_\tau^{n-2}\mathbf{B}_\tau & \dots & \mathbf{C}\mathbf{B}_\tau & \mathbf{0} \end{bmatrix}$$

$$\mathbf{V} = \mathbf{P}^*\mathbf{\Sigma}, \quad \mathbf{V}^* = \mathbf{\Xi}^{*-1}\begin{bmatrix} \mathbf{V}^+ \\ \mathbf{0} \end{bmatrix} \tag{5.10}$$

and where

$$\hat{\mathbf{J}} = \mathbf{P}_1^* \mathbf{\Phi}_\tau^{2n} \mathbf{P}_1^{*-1}, \quad \mathbf{P}^* = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{\Phi}_\tau \\ \vdots \\ \mathbf{C}\mathbf{\Phi}_\tau^{n-1} \end{bmatrix},$$

$$\mathbf{\Sigma} = [\Phi_\tau^{n-1}\mathbf{B}_\tau \ \dots \ \Phi_\tau \mathbf{B}_\tau \ \ \mathbf{B}_\tau] \tag{5.11}$$

$$\mathbf{V}^+ = \mathbf{P}_1^* \mathbf{\Phi}_\tau^n [\mathbf{\Sigma} - \mathbf{\Phi}_\tau^n \mathbf{P}_1^{*-1} \mathbf{U}_1] \tag{5.12}$$

while the nonsingular permutation matrix $\mathbf{\Xi}^* \in \mathbf{R}^{np \times np}$*, is such that*

$$\mathbf{\Xi}^* \mathbf{P}^* = \begin{bmatrix} \mathbf{P}_1^* \\ \mathbf{0} \end{bmatrix} \tag{5.13}$$

where $\mathbf{P}_1^* \in \mathbf{R}^{n \times n}$ *is the nonsingular matrix whose rows are the linearly independent rows of the matrix* $\mathbf{P}^*$*. Finally,* $\mathbf{U}_1 \in \mathbf{R}^{n \times np}$ *is the matrix containing the first* n *rows of the matrix*

$$\mathbf{U} = \mathbf{\Xi}^* \mathbf{J}_2. \tag{5.14}$$

Proof: In order to prove relation (5.7), we next generalize the approach presented in [32], to the multivariable case. More precisely, from relations (5.1) we have

$$\begin{aligned} \mathbf{y}[(\nu - n + 1)\tau] &= \mathbf{C}\xi[(\nu - n + 1)\tau] \\ \mathbf{y}[(\nu - n + 2)\tau] &= \mathbf{C}\mathbf{\Phi}_\tau \xi[(\nu - n + 1)\tau] + \mathbf{C}\mathbf{B}_\tau \mathbf{u}[(\nu - n + 1)\tau] \\ &\vdots \\ \mathbf{y}(\nu\tau) &= \mathbf{C}\mathbf{\Phi}_\tau^{n-1}\xi[(\nu - n + 1)\tau] + \textstyle\sum_{\varrho=0}^{n-2} \mathbf{C}\mathbf{\Phi}_\tau^{\varrho} \mathbf{B}_\tau \mathbf{u}[(\nu - \varrho - 1)\tau] \end{aligned}$$

or more compactly,

$$\mathbf{\Psi}(\nu\tau) = \mathbf{P}^* \xi[(\nu - n + 1)\tau] + \mathbf{J}_2 \mathbf{W}(\nu\tau) \tag{5.15}$$

where, $\mathbf{\Psi}(\nu\tau)$ and $\mathbf{W}(\nu\tau)$ are defined by (5.8) and $\mathbf{P}^*$ and $\mathbf{J}_2$ are defined by (5.11) and (5.10), respectively.

Since, by Assumption 2, the pair $(\mathbf{\Phi}_\tau, \mathbf{C})$ is observable, the matrix $\mathbf{P}^*$ has full column rank. Hence, there exists a nonsingular permutation matrix $\mathbf{\Xi}^* \in \mathbf{R}^{np \times np}$, such that relation (5.13) to hold, where, as already mentioned, $\mathbf{P}_1^* \in \mathbf{R}^{n \times n}$ is the nonsingular matrix whose rows are the linearly independent rows of the matrix $\mathbf{P}^*$. It is pointed out that matrix $\mathbf{\Xi}^*$ can be defined as a product of two nonsingular matrices $\tilde{\mathbf{\Xi}} \in \mathbf{R}^{np \times np}$ and

$\hat{\Xi} \in \mathbf{R}^{np\times np}$ via the following chain of definitions

$$\Xi^* = \overset{\sim}{\Xi}\overset{\wedge}{\Xi},\ \overset{\sim}{\Xi} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{np-n} \end{bmatrix},\ \overset{\wedge}{\Xi} = \begin{bmatrix} \mathbf{e}_{j1} \\ \mathbf{e}_{j2} \\ \vdots \\ \mathbf{e}_{jn} \\ \Xi_1^+ \end{bmatrix}$$

where $\Xi_1^+ \in \mathbf{R}^{(np-n)\times np}$ is the matrix produced by the nonsingular matrix $\Xi^+ \in \mathbf{R}^{np\times np}$ of the form

$$\Xi^+ = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_{np} \end{bmatrix}$$

by dropping the row vectors $\mathbf{e}_i$, $i = j_1, j_2, \ldots, j_n$ are the indices of the n linearly independent rows of $\mathbf{P}^*$ defined as $\mathbf{p}^*_{j_\varrho}$, $\varrho = 1, 2, \ldots, n$. Note also that $\omega_k \in \mathbf{R}^{np}$, $k = 1, 2, \ldots, np-n$ is the column vector of the form

$$\omega_k = [(\lambda_{j_1})_k \ \ (\lambda_{j_2})_k \ \ldots \ (\lambda_{j_n})_k \ \ 0 \ \ldots \ 0 \quad \underbrace{-1}_{(n+k)th\ position} \quad 0 \ \ldots \ 0]$$

where $(\lambda_{j_\varrho})_k$, $\varrho = 1, 2, \ldots, n$, $k = 1, 2, \ldots, np-n$ are the coefficients of the following dependence relation holding for the rows of the matrix $\mathbf{P}^*$

$$\sum_{\varrho=1}^{n} (\lambda_{j_\varrho})_k \mathbf{p}^{*^T}_{j_\varrho} - \mathbf{p}^{*^T}_k = 0 \ \ , \ \ k \notin \{j_1, j_2, \ldots, j_n\}$$

where, $\mathbf{p}^{*^T}_k$, $k \notin \{j_1, j_2, \ldots, j_n\}$ is the kth row of the matrix $\mathbf{P}^*$.

Now, multiplying (5.15) from the left by Ξ^*, yields

$$\mathbf{Z}^*(\nu\tau) = \begin{bmatrix} \mathbf{P}_1^* \\ 0 \end{bmatrix} \xi[(\nu - n + 1)\tau] + \mathbf{U}\mathbf{W}(\nu\tau)$$

where

$$\mathbf{Z}^*(\nu\tau) = \Xi^* \mathbf{\Psi}(\nu\tau) \tag{5.16}$$

and where $\mathbf{U}$ is defined by (5.14). Next, decompose $\mathbf{Z}^*(\nu\tau)$ and $\mathbf{U}$ as follows

$$\mathbf{Z}^*(\nu\tau) = \begin{bmatrix} \mathbf{Z}_1^*(\nu\tau) \\ \mathbf{Z}_2^*(\nu\tau) \end{bmatrix},\ \mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} \tag{5.17}$$

where $\mathbf{Z}_1^*(\nu\tau) \in \mathbf{R}^n$, $\mathbf{Z}_2^*(\nu\tau) \in \mathbf{R}^{n(p-1)}$, $\mathbf{U}_1 \in \mathbf{R}^{n\times np}$ and $\mathbf{U}_2 \in \mathbf{R}^{n(p-1)\times np}$. Clearly,

$$\mathbf{Z}_1^*(\nu\tau) = \mathbf{P}_1^*\xi[(\nu-n+1)\tau] + \mathbf{U}_1\mathbf{W}(\nu\tau). \tag{5.18}$$

From (5.18), one may easily obtain the following relation

$$\xi[(\nu-n+1)\tau] = \mathbf{P}_1^{*-1}[\mathbf{Z}_1^*(\nu\tau) - \mathbf{U}_1\mathbf{W}(\nu\tau)]. \tag{5.19}$$

Furthermore, as it can be easily shown, the following relationship holds

$$\xi[(\nu-n+1)\tau] = \mathbf{\Phi}_\tau^{2n}\xi[(\nu-3n+1)\tau] + \mathbf{\Phi}_\tau^n\mathbf{\Sigma}\mathbf{W}[(\nu-2n)\tau] + \mathbf{\Sigma}\mathbf{W}[(\nu-n)\tau] \tag{5.20}$$

where $\mathbf{W}[(\nu-n)\tau]$ and $\mathbf{W}[(\nu-2n)\tau]$ are given by (5.9), and where $\mathbf{\Sigma}$ is defined by (5.11). Introducing appropriately relation 5.19 in relation 5.20, after some algebraic manipulations, yields

$$\begin{aligned}\mathbf{Z}_1^*(\nu\tau) &= \mathbf{U}_1\mathbf{W}(\nu\tau) + \hat{\mathbf{J}}\mathbf{Z}_1^*[(\nu-2n)\tau] + \mathbf{V}^+\mathbf{W}[(\nu-2n)\tau] + \\ &+ \mathbf{P}_1^*\mathbf{\Sigma}\mathbf{W}[(\nu-n)\tau]\end{aligned} \tag{5.21}$$

where $\hat{\mathbf{J}}$ and $\mathbf{V}^+$, are defined by (5.11) and (5.12), respectively. Combining relations (5.14), (5.16)-(5.18) and (5.21), we readily obtain (5.7). This completes the proof of the Theorem. □

It is remarked at this point that matrix $\mathbf{V}$ and matrix $\mathbf{\Omega}$ are related through the following relationship

$$\mathbf{\Omega} = \mathbf{V}\hat{\mathbf{T}}\ , \quad \hat{\mathbf{T}} = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5.22}$$

Relation (5.7) will be used in the sequel for the identification of the unknown matrices $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{V}$ and $\mathbf{V}^*$. To this end, relation (5.7) is next written in the linear regression form

$$\mathbf{\Psi}(\nu\tau) = \mathbf{\Theta}\phi(\nu\tau)$$

where

$$\mathbf{\Theta} = [\mathbf{J}_1 \ \ \mathbf{J}_2 \ \ \mathbf{V} \ \ \mathbf{V}^*]$$

is the true value of the plant parameter matrix, and where

$$\phi^T(\nu\tau) = [\mathbf{\Psi}^T[(\nu-2n)\tau] \ \ \mathbf{W}^T(\nu\tau) \ \ \mathbf{W}[(\nu-n)\tau] \ \ \mathbf{W}^T[(\nu-2n)\tau]].$$

Next, define

$$\mathbf{Z}(kT_0) = [\phi(kT_0) \ \ \phi(kT_0-\tau) \ \dots \ \phi[(k-1)T_0]]$$

$$\mathbf{Y}(kT_0) = [\mathbf{\Psi}(kT_0) \;\; \mathbf{\Psi}(kT_0 - \tau) \;\; \ldots \;\; \mathbf{\Psi}[(k-1)T_0]]$$

$$\hat{\mathbf{\Theta}}(kT_0) = [\mathbf{J}_1(kT_0) \;\; \mathbf{J}_2(kT_0) \;\; \mathbf{V}(kT_0) \;\; \mathbf{V}^*(kT_0)]$$

where $\mathbf{J}_1(kT_0)$, $\mathbf{J}_2(kT_0)$, $\mathbf{V}(kT_0)$ and $\mathbf{V}^*(kT_0)]$, are the matrices $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{V}$ and $\mathbf{V}^*$ evaluated at kT_0, through the identification procedure. Clearly, the following relation holds

$$\mathbf{Y}(kT_0) = \mathbf{\Theta}\mathbf{Z}(kT_0).$$

We now chose the recursive algorithm for the estimation of $\hat{\mathbf{\Theta}}(kT_0)$ as

$$\begin{aligned}\hat{\mathbf{\Theta}}(kT_0) \;\; = \;\; & \hat{\mathbf{\Theta}}[(k-1)T_0] - [\hat{\mathbf{\Theta}}[(k-1)T_0]\mathbf{Z}[(k-1)T_0] - \mathbf{Y}[(k-1)T_0]] \times \\ & \mathbf{Z}^T[(k-1)T_0][\alpha\mathbf{I} + \mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]]^{-1} \qquad (5.23)\end{aligned}$$

where $\alpha \in \mathbf{R}^+$, $\hat{\mathbf{\Theta}}(kT_0)$ is the estimated parameter matrix $\hat{\mathbf{\Theta}}$ at time $t = kT_0$ and $\hat{\mathbf{\Theta}}_0 = \hat{\mathbf{\Theta}}(kT_0)|_{k=0}$ is arbitrarily specified. It is pointed out that the term $\alpha\mathbf{I}$ in (5.23), is added in order to avoid numerical ill conditioning, arising in the identification procedure based on the usual least-squares algorithm, when the determinant of the matrix $\mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]$ takes small values.

Commenting on the nature of the adaptive law (5.23), we point out that, it describes, as already mentioned, an on-line estimation procedure which deals with sequential data and in which the parameter estimates are recursively updated within the time-limit imposed by the sampling period T_0. It is worth noticing, at this point that, in the present case, it is presumed that, a complete block of information needed for the estimation of the plant parameters, in *not* available prior to analysis and control, as in several off-line estimation procedures. Therefore, in our case, identification and control of the plant are performed concurrently. In order to calculate the desired MRIC based decoupling controller parameters, it is necessary here to update the plant parameter estimates using (5.23) and then solve the canonical equations of Sections 3 and 4 for every time step k (see the following Subsection for details). This is in contrast, to the standard policy followed in cases where identification and control of the plant are performed separately, in which we solve equations for the plant and the controller parameters once, after an appropriate minimum number of observations (see [33]-[34] for a comparative study of the two approaches).

Note also that, the adaptive law (5.23) is chosen so that $\hat{\mathbf{\Theta}}(kT_0)$ will satisfy equation $\mathbf{Y}(kT_0) = \mathbf{\Theta}\mathbf{Z}(kT_0)$ $(k \geq 0)$ asymptotically with time, i.e., for $k \to \infty$, rather than at every time instant. In other words, in the early stages of the on-line identification procedure, the estimated parameter matrix $\hat{\mathbf{\Theta}}(kT_0)$, obtained by (5.23), is usually far from its true value Θ and it is expected that the plant parameter estimates (and consequently the controller parameter estimates) converge to their true values, only as $k \to \infty$. Therefore, exact determination of the desired MRIC based decoupling controller through the procedure presented in Sections 3 and 4, is expected here, only after a certain step of the overall control procedure. Before this step, the calculated controllers are far from being those, which guarantee the desired performance of the closed-loop system. However, it is a standard fact in all adaptive control schemes that, convergence of the parameter estimates to their true values, depends on the specific properties of the particular identification procedure used and crucially affects the adaptation. So, in what follows, we will investigate the convergence and boundedness properties of the proposed identification procedure, which are summarized in the following proposition.

Proposition 1 *Let $\tilde{\mathbf{\Theta}}(kT_0)$ be the parameter estimation error, defined as*

$$\tilde{\mathbf{\Theta}}(kT_0) = \hat{\mathbf{\Theta}}^T(kT_0) - \mathbf{\Theta}^T. \tag{5.24}$$

Then, for the parameter estimation algorithm of the form (5.23), the following properties hold

$$(a) \quad \|\hat{\mathbf{\Theta}}(kT_0)\| \leq \mu, \quad \textit{for some finite} \;\; \mu \in \mathbf{R}^+$$

$$(b) \;\; If \quad \lim_{k\to\infty} \sum_{\varrho=0}^{k} \lambda_{min}(\mathbf{Z}(\varrho T_0)\mathbf{Z}^T(\varrho T_0)) = \infty \quad then \quad \lim_{k\to\infty} \hat{\mathbf{\Theta}}(kT_0) = \mathbf{\Theta}$$

where $\lambda_{min}(\bullet)$ denotes the minimum eigenvalue of a matrix.

Proof: (a) Taking the transpose of both sides in (5.23), introducing (5.24) in the resulting relation and taking into account the fact that $\mathbf{Z}^T(kT_0)\mathbf{\Theta}^T - \mathbf{Y}^T(kT_0) = \mathbf{0}$, we readily obtain

$$\begin{aligned} \tilde{\mathbf{\Theta}}(kT_0) \;\; = \;\; & \{\mathbf{I} - [\alpha\mathbf{I} + \mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]]^{-1} \times \\ & \mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]\}\tilde{\mathbf{\Theta}}[(k-1)T_0]. \end{aligned} \tag{5.25}$$

On the basis of the Matrix Inversion Lemma, relation (5.25) may further be written as

$$\tilde{\mathbf{\Theta}}(kT_0) = \{\mathbf{I} + \frac{1}{\alpha}\mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]\}^{-1}\tilde{\mathbf{\Theta}}[(k-1)T_0]. \tag{5.26}$$

Therefore,

$$\tilde{\mathbf{\Theta}}^T(kT_0)\tilde{\mathbf{\Theta}}(kT_0) = \tilde{\mathbf{\Theta}}^T[(k-1)T_0]\{\mathbf{I}+\frac{1}{\alpha}\mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0]\}^{-2}\tilde{\mathbf{\Theta}}[(k-1)T_0]$$

$$\leq (1+\frac{\lambda_{min}(\mathbf{Z}[(k-1)T_0]\mathbf{Z}^T[(k-1)T_0])}{\alpha})^{-2}\tilde{\mathbf{\Theta}}^T[(k-1)T_0]\tilde{\mathbf{\Theta}}[(k-1)T_0] \tag{5.27}$$

By repeatedly using the above inequality, we obtain

$$\tilde{\mathbf{\Theta}}^T(kT_0)\tilde{\mathbf{\Theta}}(kT_0) \leq [\prod_{\varrho=0}^{k-1}(1+\frac{\lambda_{min}(\mathbf{Z}(\varrho T_0)\mathbf{Z}^T(\varrho T_0))}{\alpha})]^{-2}\tilde{\mathbf{\Theta}}_0^T\tilde{\mathbf{\Theta}}_0$$

$$\leq [1+\frac{1}{\alpha}\sum_{\varrho=0}^{k-1}\lambda_{min}(\mathbf{Z}(\varrho T_0)\mathbf{Z}^T(\varrho T_0))]^{-2}\tilde{\mathbf{\Theta}}_0^T\tilde{\mathbf{\Theta}}_0 \tag{5.28}$$

where $\tilde{\mathbf{\Theta}}_0 = \tilde{\mathbf{\Theta}}_0^T - \tilde{\mathbf{\Theta}}^T$. Hence, $\|\tilde{\mathbf{\Theta}}(kT_0)\|$ is uniformly bounded by $\tilde{\mathbf{\Theta}}_0$, and since $\mathbf{\Theta}$ is finite, $\hat{\mathbf{\Theta}}(kT_0)$ is also uniformly bounded by some finite $\mu \in \mathbf{R}^+$.

(b) If $\lim_{k\to\infty}\sum_{\varrho=0}^{k}\lambda_{min}(\mathbf{Z}(\varrho T_0)\mathbf{Z}^T(\varrho T_0)) = \infty$ then, from (5.27), it follows that $\lim_{k\to\infty}\tilde{\mathbf{\Theta}}(kT_0) = \mathbf{0}$, and therefore, $\lim_{k\to\infty}\hat{\mathbf{\Theta}}(kT_0) = \mathbf{\Theta}$. □

Clearly, Proposition 1 states that for the convergence of the plant parameters estimates $\hat{\mathbf{\Theta}}(kT_0)$ to their true values $\mathbf{\Theta}$ it is sufficient that the regression vector $\mathbf{Z}(kT_0)$ is persistently exciting to the amount that

$$\lim_{k\to\infty}\sum_{\varrho=0}^{k}\lambda_{min}(\mathbf{Z}(\varrho T_0)\mathbf{Z}^T(\varrho T_0)) = \infty.$$

Therefore, since adaptation and stability of the adaptive scheme depend on the convergence of the parameter estimates to their true values, it is necessary to prove excitation of $\mathbf{Z}(kT_0)$. This is done in Subsection 5.3 that follows (see Theorem 2, therein).

Remark 5.1. It is pointed out that, although controllability and observability of the sampled system (5.1) is instrumental for our analysis, no assumption is made in the present paper on the canonical structure of the triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$. This is in contrast to the standard policy of many known adaptive systems, in which controllability or observability canonical forms are assumed for the matrix triplet involved in the estimation procedure (see for example [35], [36]). The reason for not assuming here a canonical structure for the triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$ is mainly due to the fact, that canonical forms for multivariable systems are interwoven with the knowledge of a set of controllability or observability indices of the matrix triplet sought (for example, in [35], [36] a set of observability indices is needed to be known). As a consequence, when identification procedures based on canonical structures are proposed, much more prior knowledge relative to the plant is necessary as compared to our approach.

5.2 Algorithm of the synthesis of the adaptive controller

On the basis of the estimated parameter matrix $\hat{\mathbf{\Theta}}(kT_0)$ obtained by (5.23), as well as on the basis of the relations (5.4)-(5.6) and (5.22) and of anyone of the algorithms reported in the literature for the construction of a minimal realization, one can obtain the estimates needed for the computation of the unknown matrices $\hat{\mathbf{A}}_i \equiv \hat{\mathbf{A}}_i(kT_0)$, $\mathbf{\Phi}_i \equiv \mathbf{\Phi}_i(kT_0)$ and the unknown vector $\hat{\mathbf{b}}_i \equiv \hat{\mathbf{b}}_i(kT_0)$ involved in the algorithms presented in the previous Sections. Moreover, since the matrices $\mathbf{M}, \mathbf{Q}_s, \hat{\mathbf{S}}$ and $\hat{\mathbf{S}}^*$ are constructed on the basis of $\hat{\mathbf{A}}_i(kT_0), \mathbf{\Phi}_i(kT_0)$, and $\hat{\mathbf{b}}_i(kT_0)$, then provided that the matrix triplet $(\mathbf{\Phi}(kT_0), \hat{\mathbf{B}}(kT_0), \mathbf{C}(kT_0))$ is controllable and observable for any possible value of $\hat{\mathbf{\Theta}}(kT_0)$, we can obtain the following results sought:

$$\hat{\mathbf{F}} \equiv \hat{\mathbf{F}}(\hat{\mathbf{\Theta}}(kT_0)) \quad , \quad \mathbf{v} \equiv \mathbf{v}(\hat{\mathbf{\Theta}}(kT_0)) \tag{5.29}$$

whereas no update is taken otherwise.

Overall, the procedure for the synthesis of the adaptive decoupling periodic multirate-input controller, consists of the nine steps given below:

Step 1 Choose the input multiplicities of the sampling N_i such that $N_i \geq n_i$ and the sampling period τ such that $\tau = T_0/(6n-1)N$.

Step 2 Update the estimates of the matrix $\mathbf{V}$ using relation (5.23).

Step 3 Find the matrix $\mathbf{\Omega}$ using relation (5.22).

Step 4 Obtain a minimal realization for the matrix triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$ using anyone of the minimal realization algorithms reported in the literature (see e.g. the algorithms in [29]-[31]).

Step 5 Find the matrices $\hat{\mathbf{A}}_i$ and the vectors $\hat{\mathbf{b}}_i$, as well as the matrix $\mathbf{\Phi}$ using relations (5.4) and (5.5), respectively.

Step 6 Find the matrix $\mathbf{Q}_s$ on the basis of the algorithm presented in Section 3.

Step 7 Find the matrices $\hat{\mathbf{S}}$ and $\hat{\mathbf{S}}^*$ using relations (4.17) and (4.18), respectively.

Step 8 Find the matrix $\hat{\mathbf{F}}$ and the vector $\mathbf{v}$ using relations (4.21) and (4.20), respectively.

Step 9 Find the matrix $\mathbf{F}(t)$ of the periodic multirate-imput controller sought and the persistent excitation signals $v_i(t)$ using relations (4.19) and (4.2), (4.3), (4.4), respectively.

5.3 Stability analysis of the adaptive control scheme

We now investigate the stability of the closed-loop system for arbitrary initial conditions on the plant. To this end, the following fundamental result, can be established

Theorem 2 *In the closed-loop adaptive control system the regressor sequence $\phi(\nu\tau)$ is persistently exciting, i.e., there is a $\delta \geq 0$, such that*

$$\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0) = \sum_{\nu=0}^{(6n-1)N} \phi(kT_0 - \nu\tau)\phi^T(kT_0 - \nu\tau) \geq \delta\mathbf{I}. \qquad (5.30)$$

Proof: In order to prove relation (5.30), we work as follows: Set $u_i(t) = \mathbf{d}_i^T(t)\mathbf{v}_i$. Then, relation (5.7), yields

$$\begin{aligned} y_i(\nu\tau) &= \sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+i}y_i[(\nu-2n-\varrho)\tau] + \\ &+ \sum_{\kappa=1,\kappa\neq i}^{p}\sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+\kappa}y_\kappa[(\nu-2n-\varrho)\tau] + \\ &+ \sum_{j=1}^{m}\sum_{\varrho=0}^{n-2}(\mathbf{J}_2)_{(n-1)p+i,(n-\varrho-2)m+j}u_j[(\nu-\varrho-1)\tau] + \\ &+ \sum_{j=1}^{m}\sum_{\varrho=0}^{n-1}(\mathbf{V})_{(n-1)p+i,(n-\varrho-1)m+j}u_j[(\nu-n-\varrho)\tau] + \\ &+ \sum_{j=1}^{m}\sum_{\varrho=0}^{n-1}(\mathbf{V}^*)_{(n-1)p+i,(n-\varrho-1)m+j}u_j[(\nu-2n-\varrho)\tau] \qquad (5.31) \end{aligned}$$

where in general $(\mathbf{J}_1)_{rq}$, $(\mathbf{J}_2)_{rq}$, $(\mathbf{V})_{rq}$ and $(\mathbf{V}^*)_{rq}$, are the $r-q$ elements of the matrices $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{V}$ and $\mathbf{V}^*$, respectively. Introducing the pseudovariables $\beta_{i,u_j}(\nu\tau), j \in \mathbf{J}_m$ and $\beta_{i,y_\kappa}(\nu\tau), \kappa = 1, 2, \ldots, p,\ \kappa \neq i$, relation (5.31), can be decomposed as follows:

$$\beta_{i,u_j}(\nu\tau) - \sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+i}\beta_{i,u_j}[(\nu-2n-\varrho)\tau] = u_j(\nu\tau) \quad (5.32)$$

$$\begin{aligned} y_{i,u_j}(\nu\tau) &= \sum_{\varrho=0}^{n-2}(\mathbf{J}_2)_{(n-1)p+i,(n-\varrho-2)m+j}\beta_{i,u_j}[(\nu-\varrho-1)\tau] + \\ &+ \sum_{\varrho=0}^{n-1}(\mathbf{V})_{(n-1)p+i,(n-\varrho-1)m+j}\beta_{i,u_j}[(\nu-n-\varrho)\tau] + \\ &+ \sum_{\varrho=0}^{n-1}(\mathbf{V}^*)_{(n-1)p+i,(n-\varrho-1)m+j}\beta_{i,u_j}[(\nu-2n-\varrho)\tau] \quad , \\ & for \;\; j \in \mathbf{J}_m \end{aligned} \tag{5.33}$$

$$\beta_{i,y_\kappa}(\nu\tau) - \sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+i}\beta_{i,y_\kappa}[(\nu-2n-\varrho)\tau] = y_\kappa(\nu\tau) \tag{5.34}$$

$$y_{i,y_\kappa}(\nu\tau) = \sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+\kappa}\beta_{i,y_\kappa}[(\nu-2n-\varrho)\tau] \quad ,$$

$$for \;\; \kappa = 1,2,\ldots,p, \;\; \kappa \neq i \tag{5.35}$$

while

$$y_i(\nu\tau) = \sum_{j=1}^{m} y_{i,u_j}(\nu\tau) + \sum_{\kappa=1,\kappa\neq i}^{p} y_{i,y_\kappa}(\nu\tau). \tag{5.36}$$

From relations (5.32)-(5.36), we obtain

$$\begin{aligned} y_i(\nu\tau) &= (\frac{1}{p})\{\sum_{j=1}^{m}\{\sum_{\varrho=0}^{n-2}(\mathbf{J}_2)_{(n-1)p+i,(n-\varrho-2)m+j}\beta_{i,u_j}[(\nu-\varrho-1)\tau] + \\ &+ \sum_{\varrho=0}^{n-1}(\mathbf{V})_{(n-1)p+i,(n-\varrho-1)m+j}\beta_{i,u_j}[(\nu-n-\varrho)\tau] + \\ &+ \sum_{\varrho=0}^{n-1}(\mathbf{V}^*)_{(n-1)p+i,(n-\varrho-1)m+j}\beta_{i,u_j}[(\nu-2n-\varrho)\tau]\} + \\ &+ \sum_{\kappa=1,\kappa\neq i}^{p}\sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+\kappa}\beta_{i,y_\kappa}[(\nu-2n-\varrho)\tau] + \\ &+ \sum_{\kappa=1,\kappa\neq i}^{p}\{\beta_{\kappa,y_i}(\nu\tau) - \end{aligned}$$

$$- \sum_{\varrho=0}^{n-1} (\mathbf{J}_1)_{(n-1)p+\kappa,(n-\varrho-1)p+\kappa}\beta_{\kappa,y_i}[(\nu-2n-\varrho)\tau]\} \qquad (5.37)$$

whereas relation (5.32), yields

$$\begin{aligned} u_j(\nu\tau) &= (\frac{1}{p})\sum_{i=1}^{p}\{\beta_{i,u_j}(\nu\tau) - \\ &- \sum_{\varrho=0}^{n-1}(\mathbf{J}_1)_{(n-1)p+i,(n-\varrho-1)p+i}\beta_{i,u_j}[(\nu-2n-\varrho)\tau]\}. \quad (5.38)\end{aligned}$$

On the basis of relations (5.7), (5.37) and (5.38), the regressor vector $\phi(\nu\tau)$, can also be expressed as

$$\phi(\nu\tau) = \hat{\mathbf{\Sigma}}\hat{\beta}(\nu\tau)$$

where

$$\hat{\beta}^T(\nu\tau) = [\widetilde{\beta}\ (\nu\tau)\ \dots\ \widetilde{\beta}\ [(\nu-6n+2)\tau]]$$

$$\widetilde{\beta}\ (\varrho\tau) = [\widetilde{\beta}_{u_1}\ (\varrho\tau)\ \dots\ \widetilde{\beta}_{u_m}\ (\varrho\tau)\ \ \widetilde{\beta}_{y_1}\ (\varrho\tau)\ \dots\ \widetilde{\beta}_{y_p}\ (\varrho\tau)]\ ,$$

$$\varrho = \nu-6n+2,\dots,\nu$$

$$\widetilde{\beta}_{u_j}\ (\varrho\tau) = [\beta_{1,u_j}(\varrho\tau)\ \dots\ \beta_{p,u_j}(\varrho\tau)]\ ,\ \ \varrho = \nu-6n+2,\dots,\nu\ ,\ \ j \in \mathbf{J}_m$$

$$\widetilde{\beta}_{y_1}\ (\varrho\tau) = [\beta_{2,y_1}(\varrho\tau)\ \dots\ \beta_{p,y_1}(\varrho\tau)]\ ,\ \ \varrho = \nu-6n+2,\dots,\nu$$

$$\widetilde{\beta}_{y_\kappa}\ (\varrho\tau) = [\beta_{1,y_\kappa}(\varrho\tau)\ \dots\ \beta_{p-1,y_\kappa}(\varrho\tau)]\ ,$$

$$\varrho = \nu-6n+2,\dots,\nu\ ,\ \ \kappa = 2,3,\dots,p$$

and where $\hat{\mathbf{\Sigma}} \in \mathbf{R}^{(3nm+np)\times(6n-1)p(p+m-1)}$ is a full row rank matrix. Clearly, the vector $\phi(\nu\tau)$ is persistently exciting if $\hat{\beta}(\nu\tau)$ is also persistently exciting. So, in what follows, it suffices to investigate excitation of $\hat{\beta}(\nu\tau)$. To this end, observe that (5.38), can be written as

$$u_j(\nu\tau) = \psi_j^T\hat{\beta}(\nu\tau) \qquad (5.39)$$

where, $\psi_j^T \in \mathbf{R}^{(6n-1)m(p+m-1)}$ is a row vector whose elements are known. In order to prove excitation of $\hat{\beta}(\nu\tau)$, it suffices to prove that the following relationship holds

$$\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau) \geq \epsilon \mathbf{I} \tag{5.40}$$

for some $\epsilon > 0$. To this end, observe that from relation (5.39), we can easily obtain

$$\sum_{\nu=1}^{T_0/\tau} u_j^2(kT_0+\nu\tau) = \psi_j^T \{\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau)\}\psi_j. \tag{5.41}$$

Observe also that the following relation holds

$$u_j(kT_0+\nu\tau) = \begin{cases} 0, & if \ \ \nu = 1,2,\ldots,(6n-1)(N_j-n_j-1)l_j-1 \\ 1, & if \ \ \nu = (6n-1)(N_j-n_j-1)l_j,\ldots, \\ & (6n-1)(N_j-n_j)l_j-1 \end{cases}$$

Hence, relation (5.41), can also be written as

$$(6n-1)l_j + \sum_{\nu=(6n-1)(N_j-n_j)l_j}^{T_0/\tau} u_j^2(kT_0+\nu\tau) =$$

$$\psi_j^T \{\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau)\}\psi_j$$

We can then conclude that

$$\psi_j^T \{\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau)\}\psi_j \geq (6n-1)l_j$$

and that

$$\{\frac{\psi_j^T}{\|\psi_j\|}\}\{\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau)\}\{\frac{\psi_j}{\|\psi_j\|}\} \geq \frac{(6n-1)l_j}{\|\psi_j\|^2}.$$

It is now clear that, the vector $\psi_j/\|\psi_j\|$, is a vector whose norm equals to unity. Hence there is a unity norm vector such that

$$\chi^T \{\sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0+\nu\tau)\hat{\beta}^T(kT_0+\nu\tau)\}\chi - \frac{(6n-1)l_j}{\|\psi_j\|^2} \geq 0.$$

In conclusion, relation (5.40) holds. As a consequence, the vector $\hat{\beta}(\nu\tau)$ is persistently exciting. Therefore, $\phi(\nu\tau)$ is also persistently exciting and hence there is a $\delta \geq 0$ (which, in general, depends on the matrix $\hat{\mathbf{\Sigma}}$), such that relation (5.30) to hold. This completes the proof of the theorem. □

We are now able to establish the stability of the control system.

Proposition 2 *The closed-loop adaptive control system presented above is globally stable, i.e., for arbitrary finite initial conditions all states are uniformly bounded, and discrete decoupling control is asymptotically attained, i.e.* $\lim_{k\to\infty}\{\mathbf{y}(kT_0) - \mathbf{y}_M(kT_0)\} = 0$. *Furthermore, the proposed adaptive scheme provides exponential convergence of the estimated parameters.*

Proof: Since, according to Theorem 2, the regressor sequence is persistently exciting, then the difference $\hat{\mathbf{\Theta}}(kT_0) - \mathbf{\Theta}$ converges to zero. That is, the plant parameters estimates converge to their true values. As a consequence of this and of the fact that $\hat{\mathbf{\Theta}}(kT_0)$ is uniformly bounded, the controller parameter estimates (5.29) also converge to their true values. Therefore, at the sampling instants uniform boundedness of all states and $\lim_{k\to\infty}\{\mathbf{y}(kT_0) - \mathbf{y}_M(kT_0)\} = 0$ follow on the basis of (4.5) and of the fact that the ith diagonal element of the desired model is assumed to have a strictly stable denominator. Uniform boundedness of $\mathbf{u}(t)$ and $\mathbf{x}(t)$ then follows from (2.1), (4.12), (4.13) and (4.21) and from the fact that $\mathbf{w}(kT_0)$ is bounded by assumption. Finally, exponential convergence of the plant parameter estimates follows form (5.26), which together with (5.30), ensures that $\hat{\mathbf{\Theta}}(kT_0) \to \mathbf{\Theta}$ exponentially as $k \to \infty$. □

Remark 5.2. Commenting on the assumptions needed here, in order to implement the adaptive decoupling periodic multirate-input controller, we point out the following:

Assumption 1a, on the controllability and the observability of the continuous - time plant as well as on the knowledge of its order is a standard assumption in the area of adaptive control. It is worth noticing that here, controllability of the pair $(\mathbf{A}, \mathbf{B})$ is also necessary for obtaining a solution of the integral equation (3.2), with respect to the controller matrix $\mathbf{F}(t)$. Note also that, uncontrollability (and/or observability) of the pair $(\mathbf{A}, \mathbf{B})$ implies uncontrollability (and/or observability) of the plants obtained from (2.1), by discretizing with sampling periods T_0, T_N and τ. From the previous analysis, however, it becomes clear that for the implementation of the adaptive control scheme, these discretized plants must be controllable and observable.

Assumption 1b, on the knowledge of a set of LMCI indices of the $(\mathbf{A}, \mathbf{B})$, is instrumental for the implementation of the proposed adaptive scheme, since, on the one hand, the forms of the multirate controller (4.1) and the persistent excitation signals (4.2), (4.3), (4.4) depend on the LMCI used, and on the other hand, the control strategy in the case of unknown systems is based on the fundamental sampling period τ, which also depends on the knowledge of a set of LMCI. Note also that, whenever Assumption 1b is not fulfilled, one can readily compute a set of LMCI by estimating the continuous-time system matrices $\mathbf{A}$ and $\mathbf{B}$. This can be done either by using a continuous-time counterpart of the identification procedure presented in Section 5.1 or following the structural identification approach proposed in [35]. For the sake of simplicity, we assume here that the initial information about a set of LMCI of the pair $(\mathbf{A}, \mathbf{B})$ is available.

Assumption 2 on the existence of a sampling period T_0, for which controllability and observability of the matrix triplets $(\mathbf{\Phi}, \hat{\mathbf{B}}, \mathbf{C})$ and $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$ are guaranteed, is also instrumental for our analysis. In particular, observability of the pair $(\mathbf{\Phi}, \mathbf{C})$ must be guaranteed for obtaining the simple necessary and sufficient condition (3.15), for the solvability of the equation (3.13), and hence for the solvability of the decoupling control problem in the case of known systems. On the other hand, controllability and observability of the matrix triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$ is necessary for resorting to the equivalent input-output representation (5.7), for the state space system of the form (5.1), as well as for being able to apply any of the minimal realization algorithms presented in [29]-[31], which are needed here to obtain the estimates of the triplet $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$. Note that, for ensuring controllability and observability of the triplets $(\mathbf{\Phi}, \hat{\mathbf{B}}, \mathbf{C})$ and $(\mathbf{\Phi}_\tau, \mathbf{B}_\tau, \mathbf{C})$, the fundamental sampling period T_0 must be selected such that simultaneously

$$(a) \qquad \frac{2\varrho\pi j}{T_0},\ \varrho = 0, 1, \ldots, (j = \sqrt{-1}) \quad \textit{is not the difference of any two eigenvalues of the matrix } \mathbf{A}. \tag{5.42}$$

$$(b) \qquad \frac{2(6n-1)N\varrho\pi j}{T_0},\ \varrho = 0, 1, \ldots \quad \textit{is not the difference of any two eigenvalues of the matrix } \mathbf{A}. \tag{5.43}$$

$$(c) \ \ \psi(T_N) \neq 0. \tag{5.44}$$

This implies that, in the multirate adaptive case treated here, certain sampling frequencies must be avoided, as compared to the non-adaptive non-multirate case. It is pointed out that, conditions (5.42) and (5.43), are standard conditions for the selection of a regular sampling period, in order to avoid loss of controllability and observability under sampling (see [37], for a detailed analysis of this issue).

Finally, it is pointed out that, the assumption on the strict stability of the denominators of the diagonal elements $\delta_i(z)$ of the desired diagonal model, is necessary for ensuring the global stability of the proposed adaptive scheme. Without this assumption, decoupling control may be still possible but the stability of the closed-loop adaptive system cannot be guaranteed.

Remark 5.3. The results of the present paper hold also in the special case where $N_1 = N_2 = \ldots = N_m \equiv N_0$, taking into account several modifications needed in the previous analysis, in order to fit this particular case. It is important to note that in this case, less prior information is needed for the implementation of the adaptive control scheme presented above, since there is no need of the prior knowledge of a set of LMCI of the pair $(\mathbf{A}, \mathbf{B})$. We can simply take $N_0 \geq n$. With this choice, the matrices $\hat{\mathbf{S}}$ and $\hat{\mathbf{S}}^*$ have full row rank for almost all $T_{N_0} \in [T_{N_0}^1, T_{N_0}^2]$. Then, the matrix $\mathbf{F}$ and the vector $\mathbf{v}$ have the following forms

$$\hat{\mathbf{F}} = \mathbf{E}^T \begin{bmatrix} \hat{\mathbf{S}}(\hat{\mathbf{S}}\hat{\mathbf{S}}^T)^{-1}\mathbf{Q}_s \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{v} = \Delta^T \begin{bmatrix} -\hat{\mathbf{S}}^*(\hat{\mathbf{S}}^*\hat{\mathbf{S}}^{*T})^{-1}(\hat{\mathbf{A}}^n\hat{\mathbf{b}}_1 + \ldots + \hat{\mathbf{A}}^n\hat{\mathbf{b}}_m) \\ \zeta \\ \mathbf{0}_{m(N_0-m-1)} \end{bmatrix}, \quad \hat{\mathbf{A}} = \exp(\mathbf{A}T_0/N_0).$$

6 Conclusions

The discrete adaptive decoupling problem of linear time-invariant continuous-time multi-input, multi-output systems has been investigated and an indirect control scheme based on periodic multirate-input controllers has been presented. The approach proposed to solve the problem has, as compared to known related techniques, the following main advantages:

1. It reduces the solution of the problem to the solution of a simple non-homogeneous algebraic matrix equation, rather than a matrix Diophantine equation, as is needed in standard techniques.

2. It does not rely on pole-zero cancellation and hence it can be readily applied to solve the adaptive decoupling problem for nonstably invertible plants and for diagonal reference models having arbitrary poles and zeros and relative degree.

3. It is applicable in general to systems with different number of inputs and outputs, since the multirate controllers used reveal squaring down capabilities.

4. It offers a solution to the problem of ensuring persistence of excitation of the continuous-time plant without imposing any special requirement on the reference signal $\mathbf{w}(kT_0)$ (except boundedness) and

without making any assumption concerning either the existence of specific convex sets in which the estimated parameters belong or the coprimeness of the polynomials describing the ARMA model.

It is worth noticing that, in the present technique gain controllers are essentially needed to be designed, as compared to dynamic compensators needed in known techniques. Consequently, no exogenous dynamic is introduced to the control loop by our method. This improves the computational aspect of the problem, since the proposed technique does not require many on-line computations and its practical implementation requires only computer memory for storing the history of the multirate-input controller over one period of time.

The present paper gives some new insights into the adaptive decoupling problem of linear systems. The present results can be extended to solve other related adaptive control problems, as, for example, the problems of model reference adaptive control and adaptive decoupling using multirate sampled-data hold functions. Adaptive control schemes based on alternative parameter estimation algorithms (as for, example, the algorithm proposed in [36]) and without the need of persistent excitation signals are currently under investigation.

Acknowledgements

The author wishes to thank Professor C.F. Martin and the anonymous reviewers for their inspired comments and suggestions which improved the quality of the paper.

References

[1] A.B. Chammas and C.T. Leondes. On the design of linear time invariant systems by periodic output feedback, Parts I and II, *Int. J. Control*, **27** (1978), 885-903.

[2] A.B. Chammas and C.T. Leondes. Optimal control of stochastic linear systems by discrete output feedback, *IEEE Tran. Autom. Control*, **AC-23** (1978), 921-926.

[3] J.P. Greshak and C.T. Leondes. Periodically varying compensation of time-invariant systems, *Syst. Control Lett.*, **2** (1982), 88-93.

[4] P.T. Kabamba. Control of linear systems using generalized sampled-data hold functions, *IEEE Tran. Autom. Control*, **AC-32** (1987), 772-783.

[5] H.M. Al-Rahmani and G.F. Franklin. Linear periodic systems: Eigenvalue assignment using discrete periodic feedback, *IEEE Tran. Autom. Control*, **34** (1989), 99-103.

[6] P.T. Kabamba and C. Yang. Simultaneous controller design for linear time-invariant systems, *IEEE Tran. Autom. Control*, **36** (1991), 106-111.

[7] P.N. Paraskevopoulos and K.G. Arvanitis. Exact model matching of linear systems using generalized sampled-data hold functions, *Automatica*, *30* (1994), pp. 503-506.

[8] R.Ortega, P. Albertos and R. Lozano. Adaptive stabilization of discrete-time systems using linear periodically time varying controllers, *Proc. 27th IEEE C.D.C.*, Austin, Texas, December 7-9, 1988.

[9] M. Araki and T. Hagiwara. Pole assignment by multirate sampled data output feedback, *Proc. 24th IEEE Conf. Decision Contr.*, 1985, 189-193.

[10] M. Araki and T. Hagiwara. Pole assignment by multirate sampled data output feedback, *Int. J. Control*, **44** (1986), 1661-1673.

[11] T. Hagiwara and M. Araki. Design of a stable state feedback controller based on the multirate sampling of the plant output, *IEEE Trans. Autom. Control*, **AC-33** (1988), 812-819.

[12] R. Ortega and G. Kreisselmeier. Descrete-time model reference adaptive control for continuous-time systems using generalized sampled-data hold functions, *IEEE Trans. Autom. Control*, **35** (1990), 334-338.

[13] K.G. Arvanitis and P.N. Paraskevopoulos. Exact model matching of linear systems using multirate digital controllers, *Proc. 2nd E.C.C.*, vol. 3, pp. 1648-1652, Groningen, The Netherlands, June 28 - July 1, 1993.

[14] K.G. Arvanitis and P.N. Paraskevopoulos. Discrete model reference adaptive control of continuous-time linear multi-input, multi-output systems using multirate sampled-data controllers, *J. Optimization Th. Appl.*, **84** (1995), 471-493.

[15] B.S. Morgan. The synthesis of linear multivariable systems by state-variable feedback, *IEEE Trans. Autom. Control*, **AC-9** (1964), 405-411.

[16] P.L. Falb and W.A. Wolovich. Decoupling in the design and synthesis of multivariable control systems, *IEEE Trans. Autom. Control*, **AC-12** (1967), 651-659.

[17] E.G. Gilbert. The decoupling of multivariable systems by state feedback, *SIAM J. Control*, **7** (1969), 50-63.

[18] P.N. Paraskevopoulos and K.G. Arvanitis. State feedback uniform decoupling problem of linear time-varying analytic systems, *J. of the Franklin Inst.*, **330** (1993), 1135-1163.

[19] R. Johanson. *Multivariable adaptive control*, Ph.D. Thesis, TFRT-1024, Dept. of Autom. Control, Lund Institute of Technology, 1980.

[20] R.P. Singh and K.S. Nadendra. Prior information in the design of multivariable adaptive controllers, *IEEE Trans. Autom. Control*, **AC-29** (1984), 1108-4111.

[21] P.E. McDermott and D.A. Mellichamp. A decoupling pole placement self-tuning controller for a class of multivariable processes, *Optimal Control Appl. Methods*, **7** (1986), 55-79.

[22] M. Kinnaert, R. Hanus and J.L. Henrotte. A new decoupling precompensator for indirect adaptive control of multivariable linear systems, *IEEE Trans. Autom. Control*, **AC-32** (1987), 453-459.

[23] B.R.H. Wittenmark, R.H. Middleton and G.C. Goodwin. Adaptive control of multivariable systems, *Int. J. Control*, **46** (1987), 1993-2009.

[24] S.J. Lang, X.Y. Gu and T.Y. Chai. A multivariable generalized self-tuning controller with decoupling design, *IEEE Trans. Autom. Control*, **AC-31** (1986), 474-477.

[25] T.Y. Chai. A self-tuning decoupling controller for a class of multivariable systems and global convergence analysis, *IEEE Trans. Autom. Control*, **AC-33** (1988), 767-771.

[26] L. Mo and M.M. Bayoumi. Adaptive decoupling control of MIMO systems, *Int. J. Adaptive Contr. Sign. Processing*, **3** (1989), 375-393.

[27] M.M. Bayoumi and L. Mo. Consistent parameter estimation in the adaptive control for MIMO systems, *Proc. A.C.C.*, Atlanta, GA, 1988, 2002-2005.

[28] K.G. Arvanitis. Adaptive decoupling control of linear systems without a persistent excitation requirement, *J. Franklin Inst.*, **332B** (1996), 681-715.

[29] B.L. Ho and R.E. Kalman. Effective construction of linear state variable models from input/output functions, *Proc. 3rd Allerton Conf.*, pp. 449-459; Regelungstechnik, vol. 14, 1966, 545-548.

[30] L.M. Silverman. Realisation of linear dynamical systems, *IEEE Trans. Autom. Control*, **AC-16** (1971), 554-567.

[31] R. Eising and M.L.J. Hautus. Realisations algorithms for systems over a principle ideal domain, *Memorandum COSOR 78-25*, Eindhoven University of Technology, Dept. of Mathematics, Eindhoven, 1978.

[32] R. Lozano-Leal. Robust adaptive regulation without persistent excitation, *IEEE Trans. Autom. Control*, **AC-34** (1989), 1260-1267.

[33] G.C. Goodwin and K.S. Sin. *Adaptive Filtering, Prediction and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[34] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence and Robustness*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[35] R. Guidorzi. Canonical structures in the identification of multivariable systems, *Automatica*, **11** (1975), 361-374.

[36] J.K. McElven, K.R. Lee and J.E. Bennett. Identification of multivariable linear systems from input/output measurements, *IEEE Trans. Ind. Electr.*, **39** (1992), 189-193.

[37] R.E. Kalman, Y.C. Ho and K.S. Narendra. Controllability if linear dynamical systems, *Contrib. Diff. Equations*, **1** (1962), 189-213.

National Technical University of Athens, Department of Electrical and Computer Engineering, Division of Computer Science, Zographou 15773, Athens, Greece

Communicated by Clyde F. Martin