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Feedback Stabilization and Robustness of Stabilizability over Integral Domains*

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1 Introduction

In this paper, we discuss the stabilizability and its robustness for multi-input multi-output (MIMO) systems over integral domains. A criterion for the stabilizability of single-input single-output (SISO) systems modeled over integral domains was derived by Shankar and Sule [1] using ideal theory. Their approach to the stabilizability theory is called the "coordinate-free approach." Sule [2] derived a criterion for the stabilizability of MIMO systems modeled over commutative rings as well as over unique factorization domains, by introducing the notion of the "elementary factor." The robustness of stabilizability was analyzed by Shankar and Sule [1] in the case of SISO systems and by Vidyasagar *et al.* [3] in the case of MIMO systems.

In this paper, we enlarge the notion of "elementary factor" by introducing the notion of "generalized elementary factor," so that a criterion for the stabilizability is given as a generalization of Theorem 4 in [2]. We also show that if a plant is strongly stabilizable, its doubly coprime factorization (DCF) exists. These will be described in Section 3.

In the analysis of the robustness of the stabilizability for MIMO systems, we do not assume that a plant and its stabilizing controller have their right-/left-coprime factorizations as in [3]. Instead we make use of the conditions modified mainlyfrom [1] to be applicable to MIMO systems. It

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will be shown that there exist neighborhoods of a plant and its stabilizing controller modeled over an integral domain such that each element in the neighborhood of the plant is stabilized by any element in the neighborhood of the stabilizing controller. This result is applicable to MIMO n-D systems. These will be described in Section 4.

2 Preliminary

We consider the set of stable causal transfer functions as an integral domain (i.e. not including zero divisors), which is sufficiently large, in contrast with the set of stable causal transfer functions considered in [2] which can include zero divisors.

Let $\mathcal{A}$ denote an integral domain with an identity element. This domain represents the set of stable causal transfer functions. For arbitrary but fixed nonzero f in $\mathcal{A}$, $\mathcal{A}_f$ denotes the ring of fractions of $\mathcal{A}$ with respect to the set $\{f^x \,|\, x$ is any nonnegative integer$\}$. Let $\mathcal{F}$ be the field of fractions of $\mathcal{A}$, which consists of all possible transfer functions. The set of matrices of size $x \times y$ over $\mathcal{A}$, denoted by $\mathcal{A}^{x\times y}$, coincides with the set of stable causal transfer matrices. We denote by $\mathcal{A}^{x\times x*}$ the set of nonsingular square matrices of size x over $\mathcal{A}$. The set of matrices of size $x \times y$ over $\mathcal{F}$, denoted by $\mathcal{F}^{x\times y}$, coincides with all possible transfer matrices of size $x \times y$. Let $P \in \mathcal{F}^{n\times m}$ denote the transfer matrix of a plant, which has m inputs and n outputs, to be controlled. Observe that a plant P can always be represented in the form of a fraction $P = Nd^{-1}$, where N is a matrix over $\mathcal{A}$ and d is a nonzero element of $\mathcal{A}$.

We will use small letters x and y to denote arbitrary positive integers, and capital letters E and O to denote the identity matrix and the zero matrix, respectively, throughout the paper.

To define the terminology about stability of transfer matrices, we introduce a feedback system composed of a plant and a controller. Let $\widehat{F}_{ad}$ be

$$\widehat{F}_{ad} = \{(X, Y) \in \mathcal{F}^{n\times m} \times \mathcal{F}^{m\times n} \,|\, \det(E + XY) \neq 0\}, \tag{2.1}$$

and for $(P, C) \in \widehat{F}_{ad}$, let $H(P, C)$ be

$$H(P, C) = \begin{bmatrix} (E + PC)^{-1} & -P(E + CP)^{-1} \\ C(E + PC)^{-1} & (E + CP)^{-1} \end{bmatrix}. \tag{2.2}$$

The matrix $H(P, C)$ represents the transfer matrix of the feedback system (P, C) from $[\, u_1^T \quad u_2^T \,]^T$ to $[\, e_1^T \quad e_2^T \,]^T$ as shown in Fig.1.

In the following definitions, R denotes either $\mathcal{A}$ or $\mathcal{A}_f$.

Definition 2.1 R**-stabilizing controller.** If the pair $(P, C) \in \widehat{F}_{ad}$ and $H(P, C) \in R^{(m+n)\times(m+n)}$, then C is called an *R-stabilizing controller* of

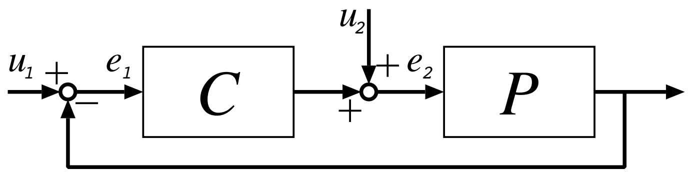

Figure 1: Feedback system

the plant P or the plant P is *R-stabilized* by C. If a plant P has an R-stabilizing controller, the plant P is said to be *R-stabilizable*. ■

Definition 2.2 R-strongly stabilizing controller. If $(P, C) \in \widehat{F}_{ad}$, $H(P, C) \in R^{(m+n)\times(m+n)}$, and $C \in R^{n\times m}$, then C is called an *R-strongly stabilizing controller* of the plant P or the plant P is *R-strongly stabilized* by C. ■

Definition 2.3 Doubly coprime factorization over R. If there exist matrices N, $\widetilde{N} \in R^{n\times m}$, $\widetilde{D} \in R^{n\times n}$, $D \in R^{m\times m}$, $X \in R^{n\times n}$, $\widetilde{X} \in R^{m\times m}$, $Y \in R^{m\times n}$, $\widetilde{Y} \in R^{m\times n}$ such that matrices D, $\widetilde{D}$, X, $\widetilde{X}$ are all nonsingular and the following equations hold:

$$P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}, \tag{2.3}$$

$$\begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ \widetilde{N} & -\widetilde{D} \end{bmatrix} \begin{bmatrix} D & Y \\ N & -X \end{bmatrix} = \begin{bmatrix} E & O \\ O & E \end{bmatrix}, \tag{2.4}$$

then (2.4) is called a doubly coprime factorization (DCF) over R of the plant P and the plant is said to have *a doubly coprime factorization (DCF) over R*. It is well known that YX^{-1} $(= \widetilde{X}^{-1}\widetilde{Y})$ becomes an R-stabilizing controller of the plant P, where X, $\widetilde{X}$, Y, and $\widetilde{Y}$ are taken from above. ■

In particular, when R is considered as $\mathcal{A}$, one may omit the phrase "$\mathcal{A}$-" or "over $\mathcal{A}$" in the above definitions.

Here we present Sule's criterion for the stabilizability of plants. (Note that only in the following proposition the symbol $\mathcal{A}$ denotes a commutative ring. Elsewhere it will denote an integral domain.)

Proposition 2.1 (Proposition 1 of [2]) *Assume that the set of stable causal transfer functions is a commutative ring $\mathcal{A}$. Let P be a strictly causal plant where the notion of strictly causal is defined as in [2]* [1]*. Then, the plant $P = Nd^{-1}$ $(N \in \mathcal{A}^{n\times m}$, $d \in \mathcal{A})$ is stabilizable if and only if there*

[1]In [2], the definition of "strictly causal" is misprinted. According to the author, the

exists a solution $X \in \mathcal{A}^{n\times n}$, $Y \in \mathcal{A}^{m\times n}$, $A \in \mathcal{A}^{n\times m}$, *and* $B \in \mathcal{A}^{m\times m}$ *for the matrix equations*

$$XN = Ad, \quad YN = Bd, \quad NY = (E - X)d. \tag{2.5}$$

Moreover, if the plant P *is stabilizable, then any stabilizing controller has the form* $C = YX^{-1}$, *where* X *and* Y *satisfy (2.5). Conversely, if (2.5) has a solution* X, Y, *then* $C = YX^{-1}$ *is causal and is a stabilizing controller.* ■

Note 2.1 The original criterion over a commutative ring in [2] requires the plant P to be strictly causal as above. However, since in our setting the set of stable causal transfer functions is an integral domain $\mathcal{A}$ rather than a general commutative ring, the strict causality in Proposition 2.1 can be relaxed according to Section 4.4 of [2]. As a result, the above proposition can be rewritten as follows:

> **Proposition 2.1′** Assume that the set of stable causal transfer functions is an integral domain $\mathcal{A}$. Then a plant P is stabilizable if and only if there exists a solution X, Y, A, and B of matrix equations (2.5) with $\det(X) \neq 0$. Moreover, if a plant P is stabilizable, any stabilizing controller has the form $C = YX^{-1}$, where matrices X and Y satisfy (2.5) and $\det(X) \neq 0$. ■

3 Stabilizability

In this section, we first present a criterion for the stabilizability over integral domains by introducing a notion of "generalized elementary factor." Then, all strongly stabilizing controllers of a given plant are characterized. We will show that if a plant is strongly stabilized by a stabilizing controller, the plant has a doubly coprime factorization (DCF).

Let us introduce some notations and symbols which will be used in the following. Let T and W be matrices such that $T = [\,N^T \quad dE\,]^T$ and $W = [\,N \quad dE\,]$, where $P = Nd^{-1} \in \mathcal{F}^{n\times m}$, $N \in \mathcal{A}^{n\times m}$, and $d \in \mathcal{A}$. Let $\mathcal{T}$ be the $\mathcal{A}$-module generated by rows of matrix T and $\mathcal{W}$ be the $\mathcal{A}$-module generated by columns of matrix W. Further, let $\mathcal{T}_f$ $\left(\mathcal{W}_f\right)$ be the $\mathcal{A}_f$-module generated by rows $\left(\text{columns}\right)$ of matrix T $\left(W\right)$. For a matrix X over R, let $I_{mR}(X)$ be the ideal in R generated by the $m \times m$ minors of

correct definition should be as follows (all symbols in the definition are as in [2]):

> DEFINITION 1. A matrix M in $\mathcal{F}$ is called causal if M has all entries in $R^{-1}\mathcal{A}$. A causal matrix M is called strictly causal if $I_t(M) \subseteq \mathcal{J}$, where $t = 1, \ldots, \min(n, m)$.

X, where R is either $\mathcal{A}$ or $\mathcal{A}_f$. We refer the readers to [4, 5] for the theory of modules.

We now introduce the notion of generalized elementary factor.

Definition 3.1 Generalized Elementary Factor. Let $\mathcal{I}$ be the sets of all m-tuples of integers $(i_1, \ldots, i_m)$ such that $1 \leq i_1 < \cdots < i_m \leq m+n$. Further let $\mathcal{J}$ be all n-tuples of integers $(j_1, \ldots, j_n)$ such that $1 \leq j_1 < \cdots < j_n \leq m+n$. Suppose that $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_n)$ are any elements of $\mathcal{I}$ and $\mathcal{J}$. Let $\Delta_{TI} \in \mathcal{A}^{m \times (m+n)}$ denote the matrix such that its (k, i_k)-entry is 1 for $1 \leq k \leq m$ and zero otherwise, $\Delta_{WJ} \in \mathcal{A}^{n \times (m+n)}$ the matrix such that its (k, j_k)-entry is 1 for $1 \leq k \leq n$ and zero otherwise. In addition, let $\mathcal{I}^* \left(\mathcal{J}^*\right)$ be the subset of $\mathcal{I} \left(\mathcal{J}\right)$ consisting of $I \in \mathcal{I}$ such that $\det(\Delta_{TI}T) \neq 0$ $\left(J \in \mathcal{J} \text{ such that } \det(\Delta_{WJ}W^T) \neq 0\right)$. We note here that matrix $\Delta_{TI}T$ $\left((\Delta_{WJ}W^T)^T\right)$ is composed of rows $i_1, \ldots, i_m$ of the matrix T $\left(\text{columns } j_1, \ldots, j_n \text{ of the matrix } W\right)$. For each $I \in \mathcal{I}^*$ and $J \in \mathcal{J}^*$, two ideals Λ_{TI} and Λ_{WJ} of $\mathcal{A}$ are defined as

$$\Lambda_{TI} = \{\lambda \in \mathcal{A} \mid \lambda T(\Delta_{TI}T)^{-1} \in \mathcal{A}^{(m+n) \times m}\}, \text{ and} \tag{3.1}$$
$$\Lambda_{WJ} = \{\lambda \in \mathcal{A} \mid \lambda W^T(\Delta_{WJ}W^T)^{-1} \in \mathcal{A}^{(m+n) \times n}\}, \tag{3.2}$$

respectively. Furthermore, ideal Λ_{IJ} is defined as

$$\Lambda_{IJ} = \Lambda_{TI} \cap \Lambda_{WJ}. \tag{3.3}$$

We denote by $\mathcal{L}_T$ $\left(\mathcal{L}_W\right)$ the set of Λ_{TI}'s for $I \in \mathcal{I}^*$ $\left(\Lambda_{WJ}\text{'s for } J \in \mathcal{J}^*\right)$ and by $\mathcal{L}$ the set of Λ_{IJ}'s for $I \in \mathcal{I}^*$ and $J \in \mathcal{J}^*$, i.e., $\mathcal{L}_T = \{\Lambda_{TI} \mid I \in \mathcal{I}^*\}$, $\mathcal{L}_W = \{\Lambda_{WJ} \mid I \in \mathcal{J}^*\}$, and $\mathcal{L} = \{\Lambda_{IJ} \mid I \in \mathcal{I}^*, J \in \mathcal{J}^*\}$. We call every element of $\mathcal{L}$ *a generalized elementary factor* of the plant P and every element of $\mathcal{L}_T$ $\left(\mathcal{L}_W\right)$ *a generalized elementary factor* of the plant P *with respect to* T $\left(W\right)$. ∎

Note 3.1 The elementary factor and the generalized elementary factor are dimensionally different, one being an element of $\mathcal{A}$ and the other an ideal of $\mathcal{A}$; the name "generalized elementary factor" is used in the sense of a generalization of the "elementary factor." When the set of stable causal transfer functions, $\mathcal{A}$, is a unique factorization domain, the generalized elementary factor Λ_{TI} $\left(\Lambda_{WJ}\right)$ with respect to T $\left(W\right)$ becomes a principal ideal and as a result its generator is the elementary factor of T $\left(W\right)$ as defined in [2]. ∎

The following proposition is a generalized version of Proposition 6 of [2]. The set of stable causal transfer functions, $\mathcal{A}$, is not restricted to a unique factorization domain and can be any integral domain.

Proposition 3.1 *Let $I \in \mathcal{I}^*$ $\left(J \in \mathcal{J}^*\right)$. (i) For each nonzero element λ_I $\left(\lambda_J\right)$ of generalized elementary factor Λ_{TI} $\left(\Lambda_{WJ}\right)$ with respect to T $\left(W\right)$ of the plant P, the $\mathcal{A}_{\lambda_I}$-module $\mathcal{T}_{\lambda_I}$ $\left(\mathcal{A}_{\lambda_J}\text{-module } \mathcal{W}_{\lambda_J}\right)$ is free of rank m $\left(n\right)$. (ii) For each nonzero element λ_{IJ} of the generalized elementary factor Λ_{IJ} of P, the $\mathcal{A}_{\lambda_{IJ}}$-module $\mathcal{T}_{\lambda_{IJ}}$ $\left(\mathcal{W}_{\lambda_{IJ}}\right)$ is free of rank m $\left(n\right)$.*

Proof: (i) Fix a nonzero $\lambda_I \in \Lambda_{TI}$. Let Δ_{TI} denote the same matrix as in Definition 3.1. Let $K = \lambda_I T(\Delta_{TI}T)^{-1}$. Then, matrix T is factorized over $\mathcal{A}_{\lambda_I}$ as

$$T = (\lambda_I^{-1}K)(\Delta_{TI}T), \tag{3.4}$$

where all entries of matrix $\lambda_I^{-1}K$ belong to $\mathcal{A}_{\lambda_I}$. Since $\lambda_I^{-1}\Delta_{TI}K$ is the identity matrix of $\mathcal{A}_{\lambda_I}^{m\times m}$ [2], the module generated by rows of matrix $\lambda_I^{-1}K$ is a free $\mathcal{A}_{\lambda_I}$-module of rank m. Every entry of matrix $\Delta_{TI}T$ is in $\mathcal{A}_{\lambda_I}$ as well as in $\mathcal{A}$. Further $\det(\Delta_{TI}T) \neq 0$ because $I \in \mathcal{I}^*$. It follows that the $\mathcal{A}_{\lambda_I}$-module $\mathcal{T}_{\lambda_I}$ is free of rank m.

Applying the same procedure as above to matrix W^T, we have an analogous result for the generalized elementary factor Λ_{WJ} with respect to W.

(ii) This is obvious from the construction of the generalized elementary factors of the plant P from Λ_{TI}'s and Λ_{WJ}'s. □

The following proposition gives the $\mathcal{A}_\lambda$-stabilizability of any plant, where λ is a nonzero element of a generalized elementary factor of a plant. The following result is independent of the stabilizability (*or* the $\mathcal{A}$-stabilizability) of the given plant.

Proposition 3.2 *Let P be any plant. Fix a generalized elementary factor Λ_{IJ} of the plant P and any nonzero element λ of Λ_{IJ}. Then the plant P has a DCF over $\mathcal{A}_\lambda$ and is $\mathcal{A}_\lambda$-stabilizable.*

Proof: Let λ be an arbitrary but fixed nonzero element of a generalized elementary factor Λ_{IJ} of the plant P. We recall that the plant P has a DCF over $\mathcal{A}$ if and only if both the $\mathcal{A}$-modules $\mathcal{T}$ and $\mathcal{W}$ are free of ranks m and n, respectively (Lemma 3 of [2]). This also holds replacing $\mathcal{A}$ by $\mathcal{A}_\lambda$, $\mathcal{T}$ by $\mathcal{T}_\lambda$, and $\mathcal{W}$ by $\mathcal{W}_\lambda$ because $\mathcal{A}_\lambda$ itself is a commutative ring and the field of fractions of $\mathcal{A}_\lambda$ coincides with $\mathcal{F}$. By Proposition 3.1, the $\mathcal{A}_\lambda$-modules

[2] $\lambda_I^{-1}\Delta_{TI}K = \lambda_I^{-1} \cdot \Delta_{TI} \cdot \lambda_I T(\Delta_{TI}T)^{-1} = \Delta_{TI}T(\Delta_{TI}T)^{-1} = I$.

$\mathcal{T}_\lambda$ and $\mathcal{W}_\lambda$ are free of ranks m and n, respectively. Therefore the plant P has a DCF over $\mathcal{A}_\lambda$. Then the $\mathcal{A}_\lambda$-stabilizability of the plant is trivial. □

From Propositions 2.1′ and 3.2, for each nonzero λ of each generalized elementary factor Λ_{IJ} of the plant P, there exist matrices X, Y, A, and B satisfying (2.5) in Proposition 2.1 over $\mathcal{A}_\lambda$ by regarding $\mathcal{A}_\lambda$ as $\mathcal{A}$. Using this fact, we present a criterion for the stabilizability in terms of generalized elementary factors as follows.

Theorem 3.1 *A plant P is stabilizable if and only if the set of generalized elementary factors of the plant P, $\mathcal{L}$, satisfies:*

$$\sum_{\Lambda_{IJ}\in\mathcal{L}} \Lambda_{IJ} = \mathcal{A}. \tag{3.5}$$

Proof: (Only If) Suppose that the plant P is stabilizable. Further, suppose that C is a stabilizing controller of P.

In the following, it is shown that the following relation holds:

$$\sum_{\Lambda_{TI}\in\mathcal{L}_T} \Lambda_{TI} = \mathcal{A}. \tag{3.6}$$

Let $P = Nd^{-1}$ and $C = N_c d_c^{-1}$, where N and N_c are matrices over $\mathcal{A}$, and d and d_c are scalars of $\mathcal{A}$. Then by Lemma 2 of [2] the direct sum of the modules generated by rows of $[\,N^T \quad dE\,]^T$ and rows of $[\,N_c^T \quad d_cE\,]^T$ is free, so that the $\mathcal{A}$-module $\mathcal{T}$ is finitely generated projective. According to Theorem 1 on p.109 of [6], there exists a finite subset F of $\mathcal{A}$ such that it generates $\mathcal{A}$ and the $\mathcal{A}_f$-module $\mathcal{T}_f$ is free for any $f \in F$. We assume, without loss of generality, that the set F does not contain zero. By this assumption, the $\mathcal{A}_f$-module $\mathcal{T}_f$ is free of rank m for all f in F.

In order to prove the relation (3.6), it suffices to show that the relation

$$\sum_{f\in F}(f^\xi) \subset \sum_{\Lambda_{TI}\in\mathcal{L}_T} \Lambda_{TI} \tag{3.7}$$

holds for a sufficiently large integer ξ since $\sum_{f\in F}(f^\xi) = \mathcal{A}$. To complete the proof, we will construct an ideal of $\mathcal{A}$ depending on f of F such that for each f in F, it is smaller than or equal to $\sum_{\Lambda_{TI}\in\mathcal{L}_T}\Lambda_{TI}$ and larger than or equal to (f^ξ). Let $f \in F$ be fixed. There are m $\mathcal{A}_f$-linearly independent elements in the $\mathcal{A}_f$-module $\mathcal{T}_f$. Let V_f be a square matrix of size m whose rows are $\mathcal{A}_f$-linearly independent elements of $\mathcal{T}_f$. We assume without loss of generality that the matrix V_f is over $\mathcal{A}$ (otherwise if V_f is a matrix over $\mathcal{A}_f$, V_f multiplied by f^x, with a sufficiently large integer x, will be a matrix over $\mathcal{A}$, so that we can regard such a matrix as "V_f."). There exist a nonnegative integer ν and a matrix $K_f \in \mathcal{A}^{(m+n)\times m}$ such that the following equation holds:

$$T = f^{-\nu} K_f V_f. \tag{3.8}$$

Then ideal $I_{m\mathcal{A}}(K_f)$ becomes the ideal we want to construct, as follows. First, we will show that

$$I_{m\mathcal{A}}(K_f) \subset \sum_{\Lambda_{TI} \in \mathcal{L}_T} \Lambda_{TI}. \tag{3.9}$$

Using the matrix Δ_{TI} in Definition 3.1, we denote matrices $\Delta_{TI}T$ and $\Delta_{TI}K_f$ by T_I and K_{fI}, respectively. Then the matrix equation $TT_I^{-1} = K_f K_{fI}^{-1}$ holds. It follows that $TT_I^{-1}\det(K_{fI}) \in \mathcal{A}^{(m+n)\times m}$, so that $\det(K_{fI}) \in \Lambda_{TI}$ from (3.1). Since $I_{m\mathcal{A}}(K_f)$ is an $\mathcal{A}$-linear combination of such $\det(K_{fI})$'s, inclusion relation (3.9) holds. Next, we show that $I_{m\mathcal{A}_f}(f^{-\nu}K_f) = \mathcal{A}_f$ since it implies that

$$(f^{\xi}) \subset I_{m\mathcal{A}}(K_f) \tag{3.10}$$

for a sufficiently large integer ξ. As a relationship between matrices T and V_f, there is an $\mathcal{A}_f$-unimodular U such that $T = U\,[\,V_f^T \quad O\,]^T$. It follows that $U = [\,f^{-\nu}K_f^T \quad Z\,]^T$ holds for some matrix Z. Hence by using Laplace's expansion, we have that $I_{m\mathcal{A}_f}(f^{-\nu}K_f) = \mathcal{A}_f$, so that (3.10) holds. It follows from (3.9) and (3.10), that the inclusion relation (3.7) holds, and as a result (3.6) also holds.

Similarly we have $\sum_{\Lambda_{WJ}\in\mathcal{L}_W} \Lambda_{WJ} = \mathcal{A}$ and hence relation (3.5) holds by the construction of the set of generalized elementary factors of the plant P, $\mathcal{L}$.

(If) To show that the plant P is stabilizable, we will construct a stabilizing controller of P from $\mathcal{A}_{\lambda_{IJ}}$-stabilizing controllers of P. According to (3.5), we select one element, denoted by λ_{IJ}, in the generalized elementary factor Λ_{IJ} for each pair $(I, J) \in \mathcal{I}^* \times \mathcal{J}^*$ such that the following equation holds:

$$\sum_{(I,J)\in\mathcal{I}^*\times\mathcal{J}^*} \lambda_{IJ} = 1. \tag{3.11}$$

In the rest of this proof, we fix λ_{IJ} for each pair (I, J) in $\mathcal{I}^* \times \mathcal{J}^*$ as in (3.11). Let $\mathcal{I}^+$ denote the set of all pairs (I, J) such that λ_{IJ} in (3.11) is nonzero. Then (3.11) can be rewritten as

$$\sum_{(I,J)\in\mathcal{I}^+} \lambda_{IJ} = 1. \tag{3.12}$$

By Proposition 3.2, the plant P has a DCF over $\mathcal{A}_{\lambda_{IJ}}$. Let $P = Nd^{-1}$, where N is a matrix over $\mathcal{A}$ and d is a scalar of $\mathcal{A}$ (note that $\mathcal{A}$, *not* $\mathcal{A}_{\lambda_{IJ}}$, is used here). Then P is $\mathcal{A}_{\lambda_{IJ}}$-stabilizable and as a result there exist matrices X_{IJ}, Y_{IJ}, A_{IJ}, and B_{IJ} over $\mathcal{A}_{\lambda_{IJ}}$ satisfying

$$X_{IJ}N = A_{IJ}d, \quad Y_{IJ}N = B_{IJ}d, \quad NY_{IJ} = (E - X_{IJ})d. \tag{3.13}$$

For any positive integer ν, there exists an a_{IJ} in $\mathcal{A}$ for each $(I,J) \in \mathcal{I}^+$ such that $\sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\nu} = 1$. Using this fact and equations in (3.13), for a sufficiently large integer ν, we have the following three matrix equations:

$$\sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}X_{IJ})N = \sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}A_{IJ})d, \tag{3.14}$$

$$\sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}Y_{IJ})N = \sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}B_{IJ})d, \tag{3.15}$$

$$\begin{aligned} N\sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}Y_{IJ}) &= \sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}(I - X_{IJ}))d \\ &= (I - \sum_{(I,J)\in\mathcal{I}^+} (a_{IJ}\lambda_{IJ}^{\nu}X_{IJ}))d, \end{aligned} \tag{3.16}$$

where all matrices of the form $\sum_{(I,J)\in\mathcal{I}^+}(\cdot)$ are over $\mathcal{A}$.

Now let $X = \sum_{(I,J)\in\mathcal{I}^+}(a_{IJ}\lambda_{IJ}^{\nu}X_{IJ})$ and $Y = \sum_{(I,J)\in\mathcal{I}^+}(a_{IJ}\lambda_{IJ}^{\nu}Y_{IJ})$. If $\det(X) \neq 0$, we immediately obtain a controller $C = YX^{-1}$ by Proposition 2.1$'$ and the proof is complete. So, in the rest of this proof, we suppose that $\det(X) = 0$ and reconstruct the matrix X to be nonsingular. The following technical results are derived in analogy with those of Lemma 4.4.21 in [7].

Let (I_0, J_0) be an arbitrary but fixed pair of $\mathcal{I}^+$. Since the plant P has a DCF over $\mathcal{A}_{\lambda_{I_0J_0}}$, there exists matrices N_0, D_0, $\widetilde{N}_0$, $\widetilde{D}_0$, Y_0, X_0, $\widetilde{Y}_0$, and $\widetilde{X}_0$ over $\mathcal{A}_{\lambda_{I_0J_0}}$ such that $P = N_0D_0^{-1} = \widetilde{D}_0^{-1}\widetilde{N}_0$ and

$$\widetilde{N}_0Y_0 + \widetilde{D}_0X_0 = E, \quad \widetilde{Y}_0N_0 + \widetilde{X}_0D_0 = E. \tag{3.17}$$

By simple calculation, it is found that for any matrix R of $\mathcal{A}_{\lambda_{I_0J_0}}^{m\times n}$, by considering I_0 as I and J_0 as J, matrices $X_{I_0J_0}$ and $Y_{I_0J_0}$ can be $(X_0 - N_0R)\widetilde{D}_0$ and $(Y_0 + D_0R)\widetilde{D}_0$, respectively, in (3.13). In the following we will construct a matrix R such that

$$X - a_{I_0J_0}\lambda_{I_0J_0}^{\nu}N_0R\widetilde{D}_0 \tag{3.18}$$

is nonsingular over $\mathcal{A}$. Having constructed such a matrix and letting $X_{I_0J_0}$ and $Y_{I_0J_0}$ be matrices $(X_0-N_0R)\widetilde{D}_0$ and $(Y_0+D_0R)\widetilde{D}_0$, respectively, we obtain a stabilizing controller of the plant P, $C = (Y+a_{I_0J_0}\lambda_{I_0J_0}^{\nu}D_0R\widetilde{D}_0)(X-a_{I_0J_0}\lambda_{I_0J_0}^{\nu}N_0R\widetilde{D}_0)^{-1}$.

It is easy to show that the following equation over $\mathcal{A}$ holds:

$$\begin{bmatrix} N & dE \\ \lambda_{I_0J_0}^{\nu}\widetilde{X}_0 & -\lambda_{I_0J_0}^{\nu}\widetilde{Y}_0 \end{bmatrix} \begin{bmatrix} Y & a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)D_0 \\ X & -a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0 \end{bmatrix} = \begin{bmatrix} dE & O \\ Z & a_{I_0J_0}\lambda_{I_0J_0}^{2\nu}\det(\widetilde{D}_0)E \end{bmatrix}, \tag{3.19}$$

where Z is a matrix over $\mathcal{A}$. The right hand side of (3.19) as well as the second matrix on the left hand side of (3.19) are nonsingular. By Laplace's expansion of the second matrix on the left hand side of (3.19), matrix $\begin{bmatrix} X & -a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0 \end{bmatrix}$ has at least one nonzero full-size minor. Let us select a nonzero full-size minor of matrix $\begin{bmatrix} X & -a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0 \end{bmatrix}$, denoted by l, having as few columns from matrix $-a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0$ as possible. Since ν is a sufficiently large integer, matrix

$$-a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0$$

is over $\mathcal{A}$, so that the full-size minor l is in $\mathcal{A}$. Suppose that the full-size minor l is obtained by excluding columns $\alpha_1,\ldots,\alpha_k$ of matrix X and including columns $\beta_1,\ldots,\beta_k$ of matrix $-a_{I_0J_0}\lambda_{I_0J_0}^{\nu}\det(\widetilde{D}_0)N_0$. Now define a matrix R_1 $(=(r_{\beta\alpha}))$ of size $m\times n$ over $\mathcal{A}_{I_0J_0}$ by

$$r_{\beta_1\alpha_1}=\cdots=r_{\beta_k\alpha_k}=1;\quad r_{\beta\alpha}=0 \text{ for all other } \alpha,\beta \tag{3.20}$$

and a matrix R by $R_1\,\mathrm{adj}\,\widetilde{D}_0$ over $\mathcal{A}_{I_0J_0}$. Following a similar discussion as in Lemma 4.4.21 of [7], we now have

$$\det(X-a_{I_0J_0}\lambda_{I_0J_0}^{\nu}N_0R\widetilde{D}_0)=\pm l, \tag{3.21}$$

which is nonzero. Therefore (3.18) is nonsingular over $\mathcal{A}$. □

In the rest of this paper, we will assume that the symbol $\mathcal{I}^+$ denotes the set of all pairs (I,J) such that λ_{IJ} in (3.11) is nonzero as in the proof of Theorem 3.1 and symbols λ_{IJ}'s with $(I,J)\in\mathcal{I}^+$ denote nonzero elements in the generalized elementary factors Λ_{IJ}'s such that $\sum_{(I,J)\in\mathcal{I}^+}\lambda_{IJ}=1$.

Note 3.2 Theorem 3.1 can be considered as a generalization of Theorem 2.1.1 in [1] concerning SISO systems. It is interesting to show how we can connect Theorem 3.1 above to Theorem 2.1.1 of [1].

Suppose that a plant $p=nd^{-1}$ with $n,d\in\mathcal{A}$. Then, we have $\Lambda_{T1}=\Lambda_{W1}=((n):d)$, $\Lambda_{T2}=\Lambda_{W2}=((d):n)$. By Theorem 3.1, letting $\boldsymbol{a}=((n):d)$ and $\boldsymbol{b}=((d):n)$, we obtain that the plant p is stabilizable if and only if $\boldsymbol{a}+\boldsymbol{b}=\mathcal{A}$ holds. This is equivalent to Theorem 2.1.1 of [1]. ■

Strong stabilization A criterion for the strong stabilizability over a principal ideal domain was given as Corollary 2.2.1 of [8]. In the following we show that even if the existence of neither right- nor left-coprime factorization of plant is assumed, the result over $\mathcal{A}$ is the same as Corollary 2.2.1 of [8]. Furthermore, when a plant does not have a DCF, the plant cannot be strongly stabilized. This will be shown after the following proposition.

Proposition 3.3 *(i) The following statements are equivalent:*

(a) A plant P is $\mathcal{A}$-strongly stabilizable.

(b) There exist matrices $N \in \mathcal{A}^{n\times m}$, $D \in \mathcal{A}^{m\times m}$, and $C \in \mathcal{A}^{m\times n}$ with $P = ND^{-1}$ such that

$$D + CN = E. \tag{3.22}$$

(c) There exist matrices $\widetilde{N} \in \mathcal{A}^{n\times m}$, $\widetilde{D} \in \mathcal{A}^{n\times n}$, and $C \in \mathcal{A}^{m\times n}$ with $P = \widetilde{D}^{-1}\widetilde{N}$ such that

$$\widetilde{D} + \widetilde{N}C = E. \tag{3.23}$$

(ii) When either (b) or (c) of (i) holds, the transfer matrix C, given above, becomes a strongly stabilizing controller of the plant P.
(iii) Conversely, for any strongly stabilizing controller C, there exist matrices N, D, $\widetilde{N}$, and $\widetilde{D}$ over $\mathcal{A}$ such that (3.22) and (3.23) hold with $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$.

Proof: We should prove (ii) and (iii) because (ii) implies "(b) or (c) $\rightarrow$ (a)" of (i) and (iii) implies "(a) $\rightarrow$ (b) and (c)" part of (i). However, we principally prove only (iii) because in the case where, for example, (3.22) holds, it is obvious that transfer matrix C in (3.22) becomes a strongly stabilizing controller of the plant P.
(iii) Let C be a strongly stabilizing controller. Let (I, J) be a pair in $\mathcal{I}^+$. Recall that symbols λ_{IJ}'s with $(I, J) \in \mathcal{I}^+$ denote nonzero elements in the generalized elementary factors Λ_{IJ}'s of the plant P such that $\sum_{(I,J)\in\mathcal{I}^+} \lambda_{IJ} = 1$.

In the following, we will construct matrices N and D satisfying (3.22) with $P = ND^{-1}$ from $\mathcal{A}_{\lambda_{IJ}}$-stabilizing controllers.

Because the plant P has a DCF over $\mathcal{A}_{\lambda_{IJ}}$ by Proposition 3.2, there exist two matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}_{\lambda_{IJ}}$ such that $\widetilde{X}D_{IJ} + \widetilde{Y}N_{IJ} = E$ where matrices N_{IJ} and D_{IJ} over $\mathcal{A}_{\lambda_{IJ}}$ satisfy $P = N_{IJ}D_{IJ}^{-1}$. In addition, obviously $EE + CO = E$ holds, where O denotes the zero matrix of size $n \times m$. From the above and by virtue of Lemma 3.1 of [3], matrix $D_{IJ} + CN_{IJ}$ is unimodular over $\mathcal{A}_{\lambda_{IJ}}$, say U_{IJ}. Let ξ be a sufficiently large integer. Then matrices $\lambda_{IJ}^{\xi} D_{IJ} U_{IJ}^{-1}$ and $\lambda_{IJ}^{\xi} N_{IJ} U_{IJ}^{-1}$ are over $\mathcal{A}$. Let a_{IJ} be an element in $\mathcal{A}$ for $(I,J) \in \mathcal{I}^+$ such that $\sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} = 1$, as in the proof of Theorem 3.1. Then we have

$$\sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} D_{IJ}U_{IJ}^{-1} + \sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} CN_{IJ}U_{IJ}^{-1} = E. \tag{3.24}$$

Here, let

$$N = \sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} N_{IJ}U_{IJ}^{-1} \text{ and } D = \sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} D_{IJ}U_{IJ}^{-1}, \tag{3.25}$$

where $N \in \mathcal{A}^{n\times m}$ and $D \in \mathcal{A}^{m\times m}$. Recall that $N_{IJ} = PD_{IJ}$ holds over $\mathcal{F}$ for each $(I,J) \in \mathcal{I}^+$. Hence we have $N = \sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} N_{IJ}U_{IJ}^{-1}$ $= \sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda_{IJ}^{\xi} PD_{IJ}U_{IJ}^{-1} = PD$ over $\mathcal{F}$, which shows $P = ND^{-1}$. Therefore (3.22) holds.

We can analogously obtain (3.23). □

Proposition 3.4 *When a plant can be strongly stabilized, then it has a DCF.*

Proof: Let us suppose that a plant P is strongly stabilized by a controller, denoted by C. Then from Proposition 3.3, we have $D + CN = E$ and $\widetilde{D} + \widetilde{N}C = E$, where matrices N, D, $\widetilde{N}$, and $\widetilde{D}$ satisfy $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$. By a simple calculation, we have

$$\begin{bmatrix} E & C \\ \widetilde{N} & -\widetilde{D} \end{bmatrix} \begin{bmatrix} D & C \\ N & -E \end{bmatrix} = \begin{bmatrix} E & O \\ O & E \end{bmatrix}, \tag{3.26}$$

which is a DCF of the plant P. □

4 Robustness of Stabilizability

In this section, we generalize the result given in Section 2.3 of [1] from the scalar case to the matrix one.

Here, plants and controllers are not restricted as in [3]. On the other hand, the topology we will use has several conditions which are mainly matrix versions of the conditions of [1]. Under our conditions, we will show that for a plant and its stabilizing controller, there exist neighborhoods of the plant and the stabilizing controller such that each element in the neighborhood of the plant is stabilized by any element in the neighborhood of the stabilizing controller (Theorem 4.1).

According to [1], we introduce two topologies $\tau_R^{x\times y}$ and $\tau^{x\times y}$. Let $\tau_R^{x\times y}$ and $\tau^{x\times y}$ be topologies on $\mathcal{A}^{x\times y}$ and on $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y}$, respectively, satisfying the conditions C0–C6 given below. We also denote by $\tau^{x\times y}$ the subspace topology on $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y^*}$ of the topology $\tau^{x\times y}$ on $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y}$. (Note that, by conditions C1 and C3 given below, $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y^*}$ becomes open in $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y}$.) The topology on the transfer matrices is defined later by using $\tau^{x\times y}$.

C0 The topological space $(\mathcal{A}^{x\times x},\tau_R^{x\times x})$ is a topological ring.

C1 Each element of $\mathcal{A}^{1\times 1}$ is closed in $\tau_R^{1\times 1}$.

C2 Let $A_1,A_2\in\mathcal{A}^{x\times y}$ and $B_1,B_2\in\mathcal{A}^{y\times y^*}$. Further let $A_1B_1^{-1}=A_2B_2^{-1}$. Suppose that $\mathcal{N}(A_1,B_1)$ is a $\tau^{x\times y}$-neighborhood of (A_1,B_1) of $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y^*}$. Then there exists a $\tau^{x\times y}$-neighborhood $\mathcal{N}(A_2,B_2)$ of (A_2,B_2) of $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y^*}$ such that for all (A_2',B_2') in $\mathcal{N}(A_2,B_2)$, there exists (A_1',B_1') in $\mathcal{N}(A_1,B_1)$ with $A_1'B_1'^{-1}=A_2'B_2'^{-1}$.

C3 The product topology $\tau_R^{x\times y}\times\tau_R^{y\times y}$ on $\mathcal{A}^{x\times y}\times\mathcal{A}^{y\times y}$ is weaker than topology $\tau^{x\times y}$ (i.e. $\tau_R^{x\times y}\times\tau_R^{y\times y}\subset\tau^{x\times y}$).

C4 The set of all units in $\mathcal{A}^{1\times 1}$ is open in $\tau_R^{1\times 1}$.

C5 The mapping from square matrices of size x to their determinants: $(\mathcal{A}^{x\times x},\tau_R^{x\times x})\to(\mathcal{A},\tau_R^{1\times 1})$ is continuous.

C6 (i) Suppose that a matrix $A\in\mathcal{A}^{x\times y}$ is partitioned as

$$A=\left[\begin{array}{c|c} A_1 & A_2\end{array}\right], \tag{4.1}$$

where matrices A_1 and A_2 have the same number of rows. Let y_1 and y_2 be the numbers of columns of matrices A_1 and A_2, respectively. Then for any $\tau_R^{x\times y}$-neighborhood $\mathcal{N}(A)$ of the matrix A, there exist some $\tau_R^{x\times y_i}$-neighborhood $\mathcal{N}(A_i)$ of the matrix A_i ($i=1$ or 2) such that

$$\mathcal{N}(A)\supset\left[\begin{array}{c|c}\mathcal{N}(A_1) & \mathcal{N}(A_2)\end{array}\right],\text{i.e.,} \tag{4.2}$$

$$\mathcal{N}(A)\supset\{\left[\begin{array}{c|c} A_1' & A_2'\end{array}\right]\mid A_1'\in\mathcal{N}(A_1),A_2'\in\mathcal{N}(A_2)\}.$$

(ii) Similarly, suppose that a matrix $A\in\mathcal{A}^{x\times y}$ is partitioned as

$$A=\left[\begin{array}{c} A_1\\ \hline A_2\end{array}\right], \tag{4.3}$$

where matrices A_1 and A_2 have the same number of columns. Let x_1 and x_2 be the numbers of rows of matrices A_1 and A_2, respectively. Then for any $\tau_R^{x\times y}$-neighborhood $\mathcal{N}(A)$ of the matrix A, there exist some $\tau_R^{x_i \times y}$-neighborhood $\mathcal{N}(A_i)$ of the matrix A_i ($i = 1$ or 2) such that

$$\mathcal{N}(A) \supset \left[\frac{\mathcal{N}(A_1)}{\mathcal{N}(A_2)}\right], \text{i.e.,} \tag{4.4}$$

$$\mathcal{N}(A) \supset \{\left[\frac{A_1'}{A_2'}\right] \mid A_1' \in \mathcal{N}(A_1), A_2' \in \mathcal{N}(A_2)\}.$$

■

From topology $\tau^{x \times y}$, we introduce a topology $\tau_q^{x \times y}$ on the set of transfer matrices of size $x \times y$; topology $\tau_q^{x \times y}$ on $\mathcal{F}^{x \times y}$ is induced as the quotient topology by the subspace topology $\mathcal{A}^{x \times y} \times \mathcal{A}^{y \times y*}$ of $\mathcal{A}^{x \times y} \times \mathcal{A}^{y \times y}$, i.e., if mapping

$$\pi : \mathcal{A}^{x \times y} \times \mathcal{A}^{y \times y*} \to \mathcal{F}^{x \times y} \tag{4.5}$$

is the natural projection, then $A \subset \mathcal{F}^{x \times y}$ is open in $\tau_q^{x \times y}$ if and only if $\pi^{-1}(A)$ belongs to $\tau^{x \times y}$.

For condition C2, we have an analogous result for Proposition 2.3.13 of [1] as follows.

Proposition 4.1 *The mapping π in (4.5) is open if and only if condition C2 holds.*

Proof: This proof can be obtained analogously to the proof of Proposition 2.3.13 of [1]. □

We give the following theorem in terms of the topology $\tau_q^{x \times y}$ on the transfer matrices under conditions C0–C6.

Theorem 4.1 *Assume that conditions C0–C6 hold. Let P be a plant and C a stabilizing controller of P. There exists a neighborhood $\mathcal{N}(P, C)$ of (P, C) in product topology $\tau_q^{n \times m} \times \tau_q^{m \times n}$ such that for any (P', C') in $\mathcal{N}(P, C)$, P' is stabilized by C'.*

Proof: In order to prove this theorem, it is sufficient to show that for any P' in $\tau_q^{n \times m}$-neighborhood $\mathcal{N}(P)$ and any C' in $\tau_q^{m \times n}$-neighborhood $\mathcal{N}(C)$, P' is stabilized by C'.

Suppose that ν is an arbitrary but fixed positive integer. Let (I, J) be a pair in $\mathcal{I}^+$. Recall once again that symbols λ_{IJ}'s with $(I, J) \in \mathcal{I}^+$ denote nonzero elements in the generalized elementary factors Λ_{IJ}'s such that $\sum_{(I,J) \in \mathcal{I}^+} \lambda_{IJ} = 1$. As in the proof of Theorem 3.1, for any positive integer ν, there exist some a_{IJ}'s in $\mathcal{A}$ for $(I, J) \in \mathcal{I}^+$ such that $\sum_{(I,J) \in \mathcal{I}^+} a_{IJ} \lambda_{IJ}^{\nu} = 1$. By conditions C0 and C4, there exists a $\tau_R^{1 \times 1}$-neighborhood $\mathcal{N}(\lambda_{IJ}^{\nu})$ of λ_{IJ}^{ν} such that for any $\lambda_{IJ}' \in \mathcal{N}(\lambda_{IJ}^{\nu})$, $\sum_{(I,J) \in \mathcal{I}^+} a_{IJ} \lambda_{IJ}'$ is a unit. We assume by condition C1, without loss of generality, that $0 \notin \mathcal{N}(\lambda_{IJ}^{\nu})$.

Since the plant P has a DCF over $\mathcal{A}_{\lambda_{IJ}}$ by Proposition 3.2, there exist matrices N, D, $\widetilde{N}$, $\widetilde{D}$, Y, X, $\widetilde{Y}$, and $\widetilde{X}$ over $\mathcal{A}$ with $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ and $C = YX^{-1} = \widetilde{X}^{-1}\widetilde{Y}$ such that

$$\begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} D & -Y \\ N & X \end{bmatrix} = U_{IJ}, \tag{4.6}$$

where all entries of matrix U_{IJ} are in $\mathcal{A}$, and matrix U_{IJ} is $\mathcal{A}_{\lambda_{IJ}}$-unimodular (note that U_{IJ} may not be $\mathcal{A}$-unimodular). Hence we assume without loss of generality that $\det(U_{IJ})$ is given as a power of λ_{IJ}. We assign the exponent to ν (that is, $\det(U_{IJ}) = \lambda_{IJ}^\nu$).

By condition C5, given a $\tau_R^{1\times 1}$-neighborhood $\mathcal{N}(\lambda_{IJ}^\nu)$ of λ_{IJ}^ν, there exists a $\tau_R^{(m+n)\times(m+n)}$-neighborhood $\mathcal{N}(U_{IJ})$ of the matrix U_{IJ} such that each $U'_{IJ} \in \mathcal{N}(U_{IJ})$ is $\mathcal{A}_{\lambda'_{IJ}}$-unimodular where λ'_{IJ} is some element of $\mathcal{N}(\lambda_{IJ}^\nu)$. Then by conditions C1 and C6, we have $\tau_R^{x\times y}$-neighborhoods $\mathcal{N}_{IJ}(N)$, $\mathcal{N}_{IJ}(D)$, $\mathcal{N}_{IJ}(-\widetilde{N})$, $\mathcal{N}_{IJ}(\widetilde{D})$, $\mathcal{N}_{IJ}(-Y)$, $\mathcal{N}_{IJ}(X)$, $\mathcal{N}_{IJ}(\widetilde{Y})$, and $\mathcal{N}_{IJ}(\widetilde{X})$ of N, D, $-\widetilde{N}$, $\widetilde{D}$, $-Y$, X, $\widetilde{Y}$, and $\widetilde{X}$, respectively, such that

$$\begin{bmatrix} \mathcal{N}_{IJ}(\widetilde{X}) & \mathcal{N}_{IJ}(\widetilde{Y}) \\ \mathcal{N}_{IJ}(-\widetilde{N}) & \mathcal{N}_{IJ}(\widetilde{D}) \end{bmatrix} \begin{bmatrix} \mathcal{N}_{IJ}(D) & \mathcal{N}_{IJ}(-Y) \\ \mathcal{N}_{IJ}(N) & \mathcal{N}_{IJ}(X) \end{bmatrix} \subset \mathcal{N}(U_{IJ}), \tag{4.7}$$

where $x \times y$ denotes the size of each matrix. We exclude singular elements from each of neighborhoods $\mathcal{N}_{IJ}(D)$, $\mathcal{N}_{IJ}(\widetilde{D})$, $\mathcal{N}_{IJ}(X)$, and $\mathcal{N}_{IJ}(\widetilde{X})$ as follows. By condition C1, the set of all nonzero elements of $\mathcal{A}$ is open in $\tau_R^{1\times 1}$. Hence by condition C5, the set of all nonsingular matrices of $\mathcal{A}^{x\times x}$, denoted by $\mathcal{O}_R^{x\times x}$, is open in $\tau_R^{x\times x}$. We regard the neighborhoods $\mathcal{N}_{IJ}(D)\cap\mathcal{O}_R^{m\times m}$, $\mathcal{N}_{IJ}(\widetilde{D})\cap\mathcal{O}_R^{n\times n}$, $\mathcal{N}_{IJ}(X)\cap\mathcal{O}_R^{n\times n}$, and $\mathcal{N}_{IJ}(\widetilde{X})\cap\mathcal{O}_R^{m\times m}$ as $\mathcal{N}_{IJ}(D)$, $\mathcal{N}_{IJ}(\widetilde{D})$, $\mathcal{N}_{IJ}(X)$, and $\mathcal{N}_{IJ}(\widetilde{X})$, respectively, which are not empty because each of them contains one of the matrices D, $\widetilde{D}$, X, and $\widetilde{X}$. Moreover, they consist of only nonsingular matrices.

In the following, we will construct a $\tau_q^{n\times m}$-neighborhood of the plant P from neighborhoods $\mathcal{N}_{IJ}(N)$ and $\mathcal{N}_{IJ}(D)$. Let $\mathcal{N}_{IJ}(N,D)$ denote the set

$$\{(N', D') \in \mathcal{A}^{n\times m} \times \mathcal{A}^{m\times m} \,|\, N' \in \mathcal{N}_{IJ}(N), D' \in \mathcal{N}_{IJ}(D)\}, \tag{4.8}$$

which is a $\tau_R^{n\times m} \times \tau_R^{m\times m}$-neighborhood. Further, let $\mathcal{N}_{IJ}(P)$ be $\pi(\mathcal{N}_{IJ}(N,D))$, which is a $\tau_q^{n\times m}$-neighborhood of the plant P by Proposition 4.1 (we use condition C2 here). We analogously obtain a $\tau_q^{m\times n}$-neighborhood of the stabilizing controller C, denoted by $\mathcal{N}_{IJ}(C)$, from neighborhoods $\mathcal{N}_{IJ}(Y)$ and $\mathcal{N}_{IJ}(X)$.

Let $\mathcal{N}(P)$ be the intersection of all such $\mathcal{N}_{IJ}(P)$'s, i.e., $\mathcal{N}(P) = \bigcap_{(I,J)\in\mathcal{I}^+} \mathcal{N}_{IJ}(P)$. Similarly, let $\mathcal{N}(C)$ be the intersection of all $\mathcal{N}_{IJ}(C)$'s, i.e., $\mathcal{N}(C) = \bigcap_{(I,J)\in\mathcal{I}^+} \mathcal{N}_{IJ}(C)$. Both $\mathcal{N}(P)$ and $\mathcal{N}(C)$ are $\tau_q^{n\times m}$-/$\tau_q^{m\times n}$-neighborhoods. In the following, we will show that for any $(P', C') \in \mathcal{N}(P) \times \mathcal{N}(C)$, P' is stabilized by C'.

For any pair $(I,J) \in \mathcal{I}^+$, for any $P' \in \mathcal{N}(P)$, and for any $C' \in \mathcal{N}(C)$, there exists an element λ'_{IJ} in the $\tau_R^{1\times 1}$-neighborhood $\mathcal{N}(\lambda_{IJ}^{\nu})$ such that P' is $\mathcal{A}_{\lambda'_{IJ}}$-stabilized by C' because of the inclusion relation in (4.7). It follows that there exist matrices $X_{IJ} \in \mathcal{A}_{\lambda'_{IJ}}^{n\times n}$, $Y_{IJ} \in \mathcal{A}_{\lambda'_{IJ}}^{m\times n}$, $A_{IJ} \in \mathcal{A}_{\lambda'_{IJ}}^{n\times m}$, and $B_{IJ} \in \mathcal{A}_{\lambda'_{IJ}}^{m\times m}$ such that

$$\begin{gathered} X_{IJ}N' = A_{IJ}d', \quad Y_{IJ}N' = B_{IJ}d', \quad N'Y_{IJ} = (E - X_{IJ})d' \\ P' = N'd'^{-1}, \quad C' = Y_{IJ}X_{IJ}^{-1} \end{gathered} \tag{4.9}$$

by Proposition 2.1′. Since $\sum_{(I,J)\in\mathcal{I}^+} a_{IJ}\lambda'_{IJ}$ is a unit of $\mathcal{A}$, for any positive integer ξ, we have $\sum_{(I,J)\in\mathcal{I}^+} b_{IJ}\lambda'^{\xi}_{IJ} = 1$ with some $b_{IJ} \in \mathcal{A}$. Let ξ be a sufficiently large integer. Then, by multiplying both left and right hand sides of first three expressions in (4.9) by λ'^{ξ}_{IJ}, their entries are all in $\mathcal{A}$. We now have

$$\sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}X_{IJ})N' = \sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}A_{IJ})d', \tag{4.10}$$

$$\sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}Y_{IJ})N' = \sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}B_{IJ})d', \tag{4.11}$$

$$\begin{aligned} N' \sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}Y_{IJ}) &= \sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}(E - X_{IJ}))d' \\ &= (E - \sum_{(I,J)\in\mathcal{I}^+} (b_{IJ}\lambda'^{\xi}_{IJ}X_{IJ}))d', \end{aligned} \tag{4.12}$$

as the summation of (4.9) for all $(I,J) \in \mathcal{I}^+$. By applying these expressions to Proposition 2.1′, we find that P' is stabilized by C' over $\mathcal{A}$ for any P' and C' in neighborhoods $\mathcal{N}(P)$ and $\mathcal{N}(C)$, respectively. □

Let us consider the similarities and differences between the conditions used in this paper and those given by Shankar and Sule [1] when $x = y = 1$. First, in [1], the topological space $(\mathcal{A}, \tau_R)$ is introduced as a topological ring, so that we consider that condition C0 was assumed implicitly in [1]. Conditions C1 and C4 are same as those in [1]. If $x = y = 1$, then conditions C2 and C3 are obviously equivalent to those in [1]. Although the condition C5 does not exist in [1], it holds naturally when $x = y = 1$. Condition C6 is effective only if $x > 1$ or $y > 1$ holds.

We now show that the robustness of the stabilizability shown above can be applied to the n-D systems. Suppose that the set of stable causal transfer functions is given as $\mathcal{A} = S^{-1}\mathbb{C}[F_{\nVdash}, \ldots, F_{\ltimes}]$, where S is the set of polynomials whose roots are not in the closed unit polydisc $\overline{U}^n = \{ z \in \mathbb{C}^n \mid z = (z_1, \ldots, z_n), |z_i| \leq 1, i = 1, \cdots, n \}$. Let the matrix norm of $\mathcal{A}^{x \times y}$, denoted by $|| \cdot ||$, be defined as

$$||A|| = \sup_{z \in \overline{U}^n} \overline{\sigma}(A(z)), \quad A \in \mathcal{A}^{x \times y}, \quad z = (z_1, \ldots, z_n), \tag{4.13}$$

where $\overline{\sigma}(A(z))$ is the maximal singular value of real matrix $A(z)$. Then if size x is equal to y, $(\mathcal{A}^{x \times x}, || \cdot ||)$ becomes a normed algebra. Let $\tau_R^{x \times y}$ be the induced topology of the matrix norm $|| \cdot ||$. Then $(\mathcal{A}^{x \times y}, \tau_R^{x \times y})$ is Hausdorff space.

Suppose that $X \in \mathcal{A}^{x \times y}$ and $Y \in \mathcal{A}^{y \times y}$. Suppose further that $X = X'H$, $Y = Y'H$, and $H \in \mathcal{A}^{y \times y}$ hold where matrices X' and Y' have no common right factors except for unimodular matrices. Let $\mathcal{N}(H)$, $\mathcal{N}(X')$, and $\mathcal{N}(Y')$ be $\tau_R^{y \times y}$-, $\tau_R^{x \times y}$-, and $\tau_R^{y \times y}$-neighborhoods of H, X', and Y', respectively. Define a neighborhood $\mathcal{N}(X, Y)$ of (X, Y) of $\mathcal{A}^{x \times y} \times \mathcal{A}^{x \times y}$ using such X', Y', and H as

$$\mathcal{N}(X, Y) = \{(X''H'', Y''H'') \,|\, H'' \in \mathcal{N}(H), X'' \in \mathcal{N}(X'), Y'' \in \mathcal{N}(Y')\}. \tag{4.14}$$

Varying matrices $X' \in \mathcal{A}^{x \times y}$ and $Y' \in \mathcal{A}^{y \times y}$ satisfying second sentence of this paragraph and further varying matrices $X \in \mathcal{A}^{x \times y}$ and $Y \in \mathcal{A}^{y \times y}$, we have the basic neighborhood of topology $\tau^{x \times y}$.

In the following, we show that the topologies $\tau_R^{x \times y}$ and $\tau^{x \times y}$ defined above satisfy conditions C0–C6.

(i) Since $(\mathcal{A}^{x \times x}, || \cdot ||)$ is a normed algebra, $(\mathcal{A}^{x \times x}, \tau_R^{x \times x})$ becomes a topological ring. Hence, condition C0 holds.

(ii) Conditions C1 and C4 are exactly same as those in [1]

(iii) Condition C2 holds Showing that condition C2 holds can be done analogously with that of [1].

(iv) Suppose that $A = A_1 A_2 \in \mathcal{A}^{x \times y}$ with $A_1 \in \mathcal{A}^{x \times y}$, $A_2 \in \mathcal{A}^{y \times y}$. Then observe that for the norm defined in (4.13), the inequality $||A|| \leq ||A_1|| \, ||A_2||$ holds even if $x \neq y$. So, for any $\tau_R^{x \times y}$-neighborhood $\mathcal{N}(A)$ of matrix A, there exist $\tau_R^{x \times y}$-neighborhood $\mathcal{N}(A_1)$ of matrix A_1 and $\tau_R^{y \times y}$-neighborhood $\mathcal{N}(A_2)$ of matrix A_2 such that

$$\mathcal{N}(A) \supset \mathcal{N}(A_1)\mathcal{N}(A_2), \tag{4.15}$$

which shows that condition C3 holds.

(v) Suppose that the determinant of a square matrix of size x is a mapping from $\mathcal{A}^{x \times x}$ to $\mathcal{A}^{1 \times 1}$, where the metric spaces of $\mathcal{A}^{x \times x}$ and $\mathcal{A}^{1 \times 1}$ are given by $|| \cdot ||$ of (4.13). Then the determinant is a continuous mapping from $(\mathcal{A}^{x \times x}, || \cdot ||)$ to $(\mathcal{A}^{1 \times 1}, || \cdot ||)$, which proves that condition C5 holds.

(vi) Suppose that $A = [\, A_1 \quad A_2 \,]$, where $A \in \mathcal{A}^{x \times (y_1 + y_2)}$, $A_1 \in \mathcal{A}^{x \times y_1}$, and $A_2 \in \mathcal{A}^{x \times y_2}$. Then observe that norm $||A_1|| \left(||A_2||\right)$ of matrix A_1 $\left(A_2\right)$

is equal to $||[A_1 \quad O]||$ $\Big(||[O \quad A_2]||\Big)$ of matrix $[A_1 \quad O]$ $\Big([O \quad A_2]\Big)$. Recall that $||A|| \leq ||[A_1 \quad O]|| + ||[O \quad A_2]||$. So, we have $||A|| \leq ||A_1|| + ||A_2||$. It follows that

$$\mathcal{N}(A) \supset [\mathcal{N}(A_1) \quad \mathcal{N}(A_2)], \tag{4.16}$$

which means that part (i) of condition C6 holds. We analogously obtain that part (ii) of the condition holds.

5 Comments

In this paper we have considered MIMO systems over integral domains. The results of the stabilizability and the robustness of stabilizability hold for MIMO systems over integral domains.

Recently, we have shown that for the criterion for Theorem 3.1, it is sufficient to consider the generalized elementary factors with respect to one matrix[9].

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